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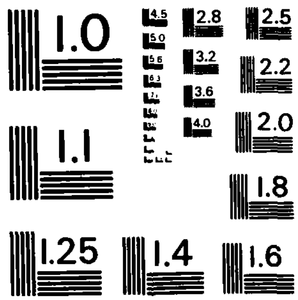
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THE AMPLITUDE DENSITY FUNCTION AND HIGH-RESOLUTION FREQUENCY ANALYSIS OF TIME SERIES

BY

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1. INTRODUCTION

There have been a number of different approaches in the scientific literature to the frequency analysis of time series, most notably the periodogram technique of Schuster (1906) and the maximum entropy method of Burg (1968). The periodogram approach was substantially improved when the fast Fourier transform algorithms became widely available (Good 1958; Cooley and Tukey 1965) as a means of speeding up the computations; this, plus the variety of smoothing algorithms proposed for improving the statistical properties of the periodogram (Daniell 1946; Blackman and Tukey 1958; Parzen 1961; Cogburn and Davis 1974; Wahba 1979), and the extension of these ideas to higher-order spectral estimation (Brillinger and Rosenblatt 1967), have together ensured the popularity, at least in the statistical literature, of periodogram-based techniques for spectrum estimation (Koopmans 1974; Brillinger 1975; Bloomfield 1976).

Concern, however, over the relatively low resolution properties of the periodogram (and also of its smoothed versions) prompted the geophysics, astronomy, and engineering fields to turn towards the higher resolution maximum entropy method of spectrum estimation (Ulrych and Bishop 1975; Kirk et al 1979), especially when it came to the analysis of short data records. For example, in Wells and Chinnery (1973), the maximum entropy method was used to separate the Chandler spectral component at approximately 0.83 cycles per year (cpy) from the annual spectral component at 1.0 cpy in short records of astronomical latitude and polar motion data. In Bolt and Currie (1975), the maximum entropy method was shown to be superior to the periodogram in terms of enhanced precision and number of torsional eigenperiods detected from data recorded at Trieste following the 1960 Chilean earthquake. The maximum entropy method was also used in Jensen and Ulrych (1973) to analyze the perturbation motions of Barnard's Star and provide detailed information concerning the number and orbital periods of its unseen companion planets.

The equivalence of the maximum entropy method with the least-squares fitting of very high order autoregressive models was pointed out by Lacoss (1971) and van den Bos (1971), and its relationship to the maximum likelihood spectrum estimator by Burg (1972). The paper of Berk (1974) is relevant here. Critics of the maximum entropy method have pointed to its sensitivity to the selection of the order of the autoregressive process, the computational complexity and programming expense of the method, its lack of a variance estimate for the resulting spectral density estimator, interpretation problems (since it is not clear what the relationship is between the "power" of the maximum entropy spectral density and the true amplitude of the spectral component at a particular frequency), and the difficulty of extending it to the frequency analysis of multiple series. We refer the interested reader to the papers by Ulrych (1972), Akaike (1969), Herring (1980), and to the references therein.

With these comments in mind, we present a different approach to the traditional methods of frequency analysis of time series. Unlike the periodogram technique and the maximum entropy method, we do not attempt to estimate the power spectrum of a series; rather, we direct attention towards the exploratory screening, identification, and resolution of any significant frequencies that might be present in a time series. For a discussion of the resolution problem, see, for example, Jenkins and Watts (1968, pp. 277-279), Brillinger (1975, p. 69), and Bloomfield (1976, pp. 96 and 172).

To accomplish this objective, we consider (in Section 2) a quantity termed the "amplitude density function" which was introduced in Siddiqui and Izenman (1981) adapted from an idea of Paul (1972), and which is derived from the spectral representation of the sum of an harmonic regression function and a stochastic error process, and on the inversion theorem associated with that representation. The amplitude density function as defined in Section 2 possesses the twin desirable properties of mean square consistency and high frequency resolution, is related to the finite Fourier transform of a tapered time series, and as such, can be computed using fast Fourier transform methods.

After using the amplitude density function to identify a (possibly, large) number of prominent frequencies in a series, the next step is to select a subset of those frequencies as input to a "hidden periodicities" regression model (see Section 3). There are basically two methods in the statistical literature for testing the significance of suspected periodicities, and both restrict themselves to the periodogram situation; that is, where the frequencies to be tested are Fourier frequencies, or, integer multiples of T^{-1}, and orthogonality relations simplify the analysis. The methods are those due to Fisher (1929, 1940) and to Hartley (1949). See also Siegel (1980). Little attempt has been made in the statistical literature, however, towards solving the more general problem of testing arbitrary non-Fourier frequencies for significance, which is the case when periods are not integral divisors of the series length. Some work in this direction can be found in Section 4.4 of Anderson (1971). In Section 3 of this paper, we discuss the use of straightforward generalizations (to the non-Fourier frequencies case) of the Fisher and Hartley tests.

We also introduce in this paper (see Section 4.3) a high-resolution "frequency trace" to investigate the extent and direction of possible nonstationarity in a time series. All the techniques described in this paper are illustrated in Section 4 by an extremely detailed frequency analysis of that perennial favorite, the annual sunspot numbers series for the period 1749-1979. Another novel feature presented in Section 4 is the use of a physically motivated data transformation of the sunspot series which dramatically improves the performance of the model described in Section 2.

2. SPECTRAL REPRESENTATION, THE INVERSION THEOREM, AND THE AMPLITUDE DENSITY FUNCTION

Let $X(t)$, $t=0,\pm1,\pm2,\ldots$, denote a time series observed at equispaced time intervals, where the origin and unit of time are chosen arbitrarily. Consider the following stochastic model:

$$X(t) = \mu(t) + \varepsilon(t), \quad t=0,\pm1,\pm2,\ldots, \tag{2.1}$$

where

$$\mu(t) = \alpha_0 + \Sigma_{j=1}^{k}\{\alpha_j\cos(2\pi f_j t) + \beta_j\sin(2\pi f_j t)\} \tag{2.2}$$

is the regression function, $\mu(t) = E\{X(t)\}$, the frequencies, f_j, $j=1,2,\ldots,k$, as well as their number, k, the long-term average α_0, and the coefficients (α_j, β_j), $j=1,2,\ldots,k$, are all assumed to be unknown constants, and the error series, $\varepsilon(t)$, $t=0,\pm1,\pm2,\ldots$, is a strictly stationary process with first two moments

$$E\{\varepsilon(t)\} = 0 \text{ and } \mathrm{cov}\{\varepsilon(t), \varepsilon(t+u)\} = c(u), \quad t,u=0,\pm1,\pm2,\ldots, \tag{2.3}$$

where

$$\Sigma_{u=-\infty}^{\infty}|c(u)| < \infty. \tag{2.4}$$

Under condition (2.4), the second-order spectral density function

$$g_\varepsilon(f) = \Sigma_{u=-\infty}^{\infty}e^{2\pi ifu}c(u), \; -\tfrac{1}{2} \le f \le \tfrac{1}{2}\,, \tag{2.5}$$

of the error series is bounded and uniformly continuous. While α_0 and the coefficients (α_j, β_j), $j=1,2,\ldots,k$, enter the model (2.1)-(2.2) in a linear fashion, the frequencies f_j, $j=1,2,\ldots,k$, (and k) enter in a non-linear way. Thus, if k and the frequencies f_j, $j=1,2,\ldots,k$, were known *à priori*, the coefficients could be estimated by standard regression techniques.

2.1. The Amplitude Density Function

For convenience, replace the trigonometric functions in (2.2) by

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \text{ and } \cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}),$$

and write $\gamma_j = \alpha_j - i\beta_j$. Denote also by $\bar{\gamma}_j$ the complex congugate of γ_j, so that $|\gamma_j|^2 = \gamma_j\bar{\gamma}_j = \alpha_j^2 + \beta_j^2$ is the squared amplitude of the complex number γ_j. Then, (2.2) becomes

$$\mu(t) = \alpha_0 + \Sigma_{j=1}^{k}\{\gamma_j e^{2\pi i f_j t} + \bar{\gamma}_j e^{-2\pi i f_j t}\}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i f t}\, dA_\mu(f), \qquad (2.6)$$

where the function $A_\mu(f)$ is defined in terms of its differential increments, namely,

$$dA_\mu(0) = \alpha_0, \quad dA_\mu(f_j) = \gamma_j/2, \quad dA_\mu(-f_j) = \bar{\gamma}_j/2, \quad j=1,2,\ldots,k, \qquad (2.7)$$

and $dA_\mu(f) = 0$ for all other f in $[-\frac{1}{2}, \frac{1}{2}]$. The Fourier-Stieltjes integral on the right-hand-side of (2.6) is the spectral representation of the regression function $\mu(t)$ in (2.2), and the function $A_\mu(f)$, $-\frac{1}{2} \le f \le \frac{1}{2}$, in (2.6) is of bounded variation.

Furthermore, by a theorem of Cramér (1942), the error series in (2.1) has a spectral representation given by

$$\varepsilon(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi ift}\, dA_\varepsilon(f), \tag{2.8}$$

where $A_\varepsilon(f)$ is a complex-valued stochastic process having orthogonal increments; that is, $E\{dA_\varepsilon(f)\} = 0$, and

$$E\{dA_\varepsilon(f)\overline{dA_\varepsilon(f')}\} = \delta\{f-f'\}g_\varepsilon(f)df, \quad -\tfrac{1}{2} \leq f \leq \tfrac{1}{2}, \tag{2.9}$$

where $\delta\{\alpha\}$ is the Kronecker delta ($\delta\{\alpha\}=1$ if $\alpha=0$, and is 0 otherwise).

Combining (2.6) and (2.8), the spectral representation of the $X(t)$-process is, therefore, given by

$$X(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi ift}\, dA_X(f), \tag{2.10}$$

where $A_X(f) = A_\mu(f) + A_\varepsilon(f)$. In this representation, $A_X(f)$ has jumps at $f = \pm f_j$, $j=1,2,\ldots,k$, and at $f = 0$, and is stochastically continuous at all other points. We shall henceforth call the process $A_X(f)$, $-\frac{1}{2} \leq f \leq \frac{1}{2}$, in (2.10) the <u>(complex-valued) amplitude process corresponding to</u> $X(t)$, $t=0,\pm1,\pm2,\ldots$, or, just the <u>amplitude process</u> if the series is understood.

Consider now a partition of the frequency band $(0, \frac{1}{2})$ into a number of nonoverlapping subbands of equal length $\Delta f > 0$, and let f be an arbitrary frequency in $(0, \frac{1}{2})$ such that

$$0 < f-\tfrac{1}{2}\Delta f < f+\tfrac{1}{2}\Delta f < \tfrac{1}{2}. \tag{2.11}$$

Define

$$\Delta A_X(f) = A_X(f+\tfrac{1}{2}\Delta f) - A_X(f-\tfrac{1}{2}\Delta f) = \Delta A_\mu(f) + \Delta A_\varepsilon(f) \tag{2.12}$$

to be the increment of the amplitude process in the frequency subband $(f-\frac{1}{2}\Delta f,\ f+\frac{1}{2}\Delta f)$ of length Δf around f. Now, if we shrink the value of Δf sufficiently, it will eventually become smaller than $\min_{1 \leq m \neq n \leq k}(|f_m - f_n|)$. For such a Δf, at most one of the k frequencies in the model (2.2) will fall into an interval of length Δf on the frequency axis. It follows that, for Δf small enough,

$$\Delta A_\mu(f) = \begin{cases} 0, \text{ if no } f_j \text{ is in } (f-\frac{1}{2}\Delta f,\ f+\frac{1}{2}\Delta f) \\ \frac{1}{2}\gamma_j, \text{ if there is an } f_j \text{ in } (f-\frac{1}{2}\Delta f,\ f+\frac{1}{2}\Delta f) \end{cases}, \quad j=1,2,\ldots,k, \tag{2.13}$$

and

$$\Delta A_\varepsilon(f) = g_\varepsilon(f')df, \tag{2.14}$$

where f' is in $(f-\frac{1}{2}\Delta f,\ f+\frac{1}{2}\Delta f)$. Thus, as $\Delta f \downarrow 0$,

$$\Delta A_X(f) \to 0 \text{ and } \Delta A_X(f)/\Delta f \to g_\varepsilon(f), \quad \text{if } f \neq f_j,\ j=1,2,\ldots,k, \tag{2.15}$$

and

$$\Delta A_X(f) \to \frac{1}{2}\gamma_j \text{ and } \Delta A_X(f)/\Delta f \to \infty, \quad \text{if } f = f_j,\ j=1,2,\ldots,k. \tag{2.16}$$

The following inversion theorem allows us to express $\Delta A_X(f)$ in terms of the series $X(t)$, $t=0,\pm1,\pm2,\ldots$.

Theorem 1. If $f \pm \frac{1}{2}\Delta f$ are continuity points of $A_\mu(f)$, and hence also of $A_X(f)$, then

$$\Delta A_X(f) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{t=-N}^{N} \frac{\sin(\pi t\Delta f)}{\pi t} e^{-2\pi i f t} X(t), \quad -\tfrac{1}{2} < f < \tfrac{1}{2}, \tag{2.17}$$

where, for $t=0$, the term in the summand is $X(0)\Delta f$, and the limit is taken in the mean-square sense.

<u>Proof</u>. See Doob (1953), Theorem 4.1 of Chapter X, or Hannan (1970), Theorem 3" of Chapter II.

This theorem, therefore, provides us with a natural estimator of $\Delta A_X(f)$. Let T be the number of data values in the time series, and let N be such that $2N+1 \leq T$, but of the order of T. Take the time-origin to be the $(N+1)$<u>st</u> value, and write

$$\Delta A_X^{(T)}(f) = \Sigma_{t=-N}^{N} \frac{\sin(\pi t \Delta f)}{\pi t} e^{-2\pi i f t} X(t), \quad -\tfrac{1}{2} < f < \tfrac{1}{2}. \qquad (2.18)$$

Note that, for real-valued X(t),

$$\overline{\Delta A_X^{(T)}(f)} = \Delta A_X^{(T)}(-f) \quad \text{and} \quad \Delta A_X^{(T)}(f+1) = \Delta A_X^{(T)}(f),$$

so that, without loss of generality, we may take the principal domain of the estimator (2.18) to be the frequency band $0 < f < \tfrac{1}{2}$.

There are alternative ways of visualizing $\Delta A_X^{(T)}(f)$. The finite Fourier transform of the series X(t), $t=0,\pm1,\pm2,\ldots,\pm N$, is given by $\Sigma_{t=-N}^{N} e^{-2\pi i f t} X(t)$, $0 < f < \tfrac{1}{2}$, which, when integrated between the frequencies $f-\tfrac{1}{2}\Delta f$ and $f+\tfrac{1}{2}\Delta f$, equals

$$\begin{aligned}
\int_{f-\frac{1}{2}\Delta f}^{f+\frac{1}{2}\Delta f} [\, \Sigma_{t=-N}^{N} e^{-2\pi i f t} X(t) \,] \, df
&= \Sigma_{t=-N}^{N} [\, \int_{f-\frac{1}{2}\Delta f}^{f+\frac{1}{2}\Delta f} e^{-2\pi i f t} \, df \,] X(t) \\
&= \Sigma_{t=-N}^{N} \frac{1}{2\pi i t} (e^{i\pi t \Delta f} - e^{-i\pi t \Delta f}) e^{-2\pi i f t} X(t) \\
&= \Delta A_X^{(T)}(f),
\end{aligned}$$

so that, if Δf is taken to be small (relative to 1/T), then $\Delta A_X^{(T)}(f)$ in (2.18) will be essentially proportional to the finite Fourier transform of X(t) at frequency f $(0 < f < \tfrac{1}{2})$. The factor $h(t) = \sin(\pi t \Delta f)/\pi t$ in (2.18) has maximum value Δf (at $t=0$), and will be very close to Δf for all other values of t. In fact, if $T=231$ and $\Delta f = 10^{-4}$ (see Section 4), then the minimum value of $h(t)$ is 0.9998×10^{-4} (at $t=\pm 115$), and the relative variation of that function is, therefore, (maximum-minimum)/Δf $< 10^{-3}$, or, 0.1 percent. Paul (1972) had a similar idea of using the integral of the finite Fourier transform of $X(t)$, but did little more than dwell at length on the inversion formula (2.17).

Furthermore, for a given value of Δf, (2.18) may be regarded as the finite Fourier transform of the "tapered" series $Y(t)=h(t)X(t)$, $t=0,\pm 1,\pm 2,\ldots,\pm N$, where $(\Delta f)^{-1}h(t)$ is related to the Riemann-Lanczos convergence factor for Fourier transforms; see Lanczos (1956, Chapter IV) and Brillinger (1975, Section 3.3). If we let $H^{(T)}(f)=\Sigma_{t=-N}^{N} e^{-2\pi i f t} h(t)$ be the finite Fourier transform of $h(t)$, then, for Δf very small, it is easy to show that $H^{(T)}(f) \doteq (\Delta f)\sin(\pi f T)/\sin(\pi f)$, which, as a function of f, is concentrated around $f=0$ with maximum value $T\Delta f$, and fluctuates in sign for $0 < f \leq \frac{1}{2}$; see Brillinger (1975, p. 51).

Consider now the statistic

$$a_X^{(T)}(f) = |\Delta A_X^{(T)}(f)|/\Delta f \tag{2.19}$$

as an estimator of the (real-valued) function

$$a_X(f) = |\Delta A_X(f)|/\Delta f \ , \tag{2.20}$$

for $0 < f < \frac{1}{2}$. We call the statistic (2.19) the <u>amplitude density function</u> of the values $X(t)$, $t=0,\pm 1,\pm 2,\ldots,\pm N$. In doing this, we differ from Paul (1972), who used the same terminology for the quantity (2.17). As an immediate consequence of the above theorem, we have the following

Corollary. In the limit, as $T \to \infty$, $a_X^{(T)}(f)$ is a mean square consistent estimator of $a_X(f)$; that is,

$$\lim_{T\to\infty} E\{|a_X^{(T)}(f) - a_X(f)|^2\} = 0, \quad 0 < f < \tfrac{1}{2}. \tag{2.21}$$

Proof. Use the fact that (mean square) convergence of a sequence of complex numbers to a complex number implies (mean square) convergence of the real and imaginary parts of the former to the corresponding real and imaginary parts of the latter.

For practical applications, attention will usually center on the user's choice of Δf when computing $a_X^{(T)}(f)$, $0 < f < \tfrac{1}{2}$, in (2.19). Dividing the frequency interval $(0,\tfrac{1}{2})$ into disjoint subintervals each of length Δf means that just under $1/2\Delta f$ points need to be computed to obtain the complete graph of $a_X^{(T)}(f)$, $0 < f < \tfrac{1}{2}$. Thus, for example, if $\Delta f = 10^{-4}$ (see Section 4), then we need to compute approximately 5000 values. In general, the choice of Δf will be tied to the length of the series, and to the spectral distribution of the k frequencies. Typically, the longer the series and the larger the number of discrete frequencies to be recovered, the smaller the value of Δf that should be used. One procedure that works quite well if the series is long and if little is known concerning its frequency structure is to use a sequence of decreasing values of Δf which provide, in turn, an increasing degree of frequency resolution and smoothness of the graph of $a_X^{(T)}(f)$, $0 < f < \tfrac{1}{2}$; see Siddiqui and Izenman (1981). Generally, a choice of $\Delta f = 10^{-4}$ seems to give useful results for a wide variety of series lengths; see the discussion in Section 4.3 of this paper. For especially long series lengths, $\Delta f = 10^{-5}$ may be preferred.

2.2. Frequency Resolution Properties

We next address ourselves to the frequency resolution properties of the amplitude density function (2.19). Let Δf be fixed, but smaller than $\min_{1\leq m\neq n\leq k}(|f_m-f_n|)$. Since, as a function of f, $a_X(f)$ in (2.20) possesses local maxima at the k points $f=f_j$, $j=1,2,\ldots,k$, the locations of the k largest relative maxima of $a_X^{(T)}(f)$ (out of, say, the $m^* \geq k$ local maxima actually obtained) provide us with appropriate estimates of the f_j, $j=1,2,\ldots,k$. The resolution properties of $a_X^{(T)}(f)$ then follow from the fact that

$$[a_X^{(T)}(f)]^2 \leq |\Sigma_{t=-N}^{N} e^{-2\pi i f t} X(t)|^2 , \tag{2.22}$$

and from published results on the asymptotic properties of periodogram estimates of the f_j, $j=1,2,\ldots,k$, derived under a variety of assumptions on the error series. Within the context of the present paper, we consider the following assumption.

(A) <u>The series</u> $\varepsilon(t)$, $t=0,\pm1,\pm2,\ldots$, <u>in (2.1) is assumed to be a sequence of independently and identically distributed (i.i.d.) random variables (a pure noise series) each with mean zero and finite variance</u>; that is, in (2.3) we set $c(u) = 0$ for $u \neq 0$, while (2.4) becomes $c(0) < \infty$. See Whittle (1952), Walker (1971), and more recently Damsleth and Spjøtvoll (1982).

In the following, we write $\underset{\sim}{f} = (f_1,f_2,\ldots,f_k)$ and $\underset{\sim}{f}^{(T)} = (f_1^{(T)},f_2^{(T)},\ldots,f_k^{(T)})$, where the $f_j^{(T)}$, $j=1,2,\ldots,k$, are the k largest local maxima of $a_X^{(T)}(f)$, $0 < f < \frac{1}{2}$, and set

$$\phi_X^{(T)}(\underset{\sim}{f}) = \Sigma_{j=1}^{k} a_X^{(T)}(f_j) \quad . \tag{2.23}$$

Then, subject to an appropriate separation condition imposed on the f_j, $j=1,2,\ldots,k$, such that

$$\lim_{T\to\infty} \min_{1\leq m\neq n\leq k}(T|f_m - f_n|) = \infty \tag{2.24}$$

(which is needed to ensure that two different frequency estimates do not converge in probability to the same value), $\phi_X^{(T)}(\underset{\sim}{f})$ in (2.23) is maximized when $\underset{\sim}{f} = \underset{\sim}{f}^{(T)}$. We shall also write $f_{j,0}$, $\alpha_{j,0}$, and $\beta_{j,0}$ for the true values of f_j, α_j, and β_j respectively, $j=1,2,\ldots,k$, and without loss of generality, we set $\alpha_0 = 0$.

Theorem 2. Let $X(t)$, $t=0,\pm1,\pm2,\ldots$, be a time series satisfying (2.1) and (2.2), where the error series $\varepsilon(t)$ satisfies assumption (A), and (2.24) holds. Then, $f_j^{(T)} - f_{j,0} = o_p(T^{-1})$, $j=1,2,\ldots,k$; that is, $f_j^{(T)}$ converges in probability to $f_{j,0}$ as $T\to\infty$, $j=1,2,\ldots,k$.

Proof. It suffices to consider the function

$$\psi_X^{(T)}(\underset{\sim}{f}) = \Sigma_{j=1}^{k} b_X^{(T)}(f_j) \quad , \tag{2.25}$$

where $b_X^{(T)}(f) = (\Delta f)^2[a_X^{(T)}(f)]^2$, $0 < f < \frac{1}{2}$. Set $\gamma_{\ell,0} = \alpha_{\ell,0} - i\beta_{\ell,0}$, $\ell=1,2,\ldots,k$, and $H^{(T)}(u) = \Sigma_{t=-N}^{N} h(t)e^{-2\pi iut}$, $0 < u < \frac{1}{2}$. Then, following the methods of Walker (1971), we have that

$$b_X^{(T)}(f_j) =$$

$$= |\Sigma_{t=-N}^{N} h(t)e^{-2\pi i f_j t}[\Sigma_{\ell=1}^{k}\{\gamma_{\ell,0}e^{2\pi i f_{\ell,0}t}+\bar{\gamma}_{\ell,0}e^{-2\pi i f_{\ell,0}t}\}+\varepsilon(t)]|^2$$

$$= |\Sigma_{\ell=1}^{k}\{\gamma_{\ell,0}H^{(T)}(f_j-f_{\ell,0})+\bar{\gamma}_{\ell,0}H^{(T)}(f_j+f_{\ell,0})\} +$$

$$+\Sigma_{t=-N}^{N} h(t)e^{-2\pi i f_j t}\varepsilon(t)|^2$$

$$= |\Sigma_{\ell=1}^{k}\{\gamma_{\ell,0}H^{(T)}(f_j-f_{\ell,0})+\bar{\gamma}_{\ell,0}H^{(T)}(f_j+f_{\ell,0})\}|^2 +$$

$$+|\Sigma_{t=-N}^{N}h(t)e^{-2\pi i f_j t}\varepsilon(t)|^2+2\mathrm{Re}\{(\Sigma_{t=-N}^{N}h(t)e^{-2\pi i f_j t}\varepsilon(t))\times$$

$$\times[\Sigma_{\ell=1}^{k}(\bar{\gamma}_{\ell,0}H^{(T)}(f_j-f_{\ell,0})+\gamma_{\ell,0}H^{(T)}(f_j+f_{\ell,0}))]\} \quad . \qquad (2.26)$$

Since the variances of the real and imaginary parts of $\Sigma_{t=-N}^{N}h(t)e^{-2\pi i f_j t}\varepsilon(t)$ are each respectively $\frac{1}{2}c(0)(\Delta f)^2T+O(1)$, then

$$\Sigma_{t=-N}^{N} h(t)e^{-2\pi i f_j t}\varepsilon(t) = O_p(T^{\frac{1}{2}}) \ .$$

There are k^2 differences in (2.26) of the form $f_j-f_{\ell,0}$, $\ell=1,2,\ldots,k$, $j=1,2,\ldots,k$; however, according to the separation condition (2.24), only k of these differences can be $O(T^{-1})$, namely, $f_j-f_{j,0}$, $j=1,2,\ldots,k$.

Now, for small values of u, the function $|H^{(T)}(u)|^2 \doteq (\Delta f)^2\sin^2(\pi uT)/\sin^2(\pi u)$ decreases monotonically from its absolute maximum of (approximately) $(\Delta fT)^2$ to a minimum of zero at $u=T^{-1}$. Hence, if $f_j-f_{j,0}$ is small enough,

$b_X^{(T)}(f_j)$ is dominated by the term $|\gamma_{j,0}H^{(T)}(f_j-f_{j,0})|^2$; in fact,

$$b_X^{(T)}(f_{j,0}) = |\gamma_{j,0}|^2(\Delta f)^2T^2 + o_p(T^{3/2}) \ , \tag{2.27}$$

and if $\{S_T\}$ is a sequence of sets in $\underset{\sim}{f}$-space for which (2.24) holds, then,

$$\max_{\underset{\sim}{f}\in S_T} \{|\psi_X^{(T)}(\underset{\sim}{f}) - \Sigma_{j=1}^k(\alpha_{j,0}^2+\beta_{j,0}^2)|H^{(T)}(f_j-f_{j,0})|^2\} = o_p(T^{3/2}) \ . \tag{2.28}$$

If we now let

$$R_T(\delta) = \{\underset{\sim}{f}: |f_j-f_{j,0}| < \delta/T, \ \ j=1,2,\ldots,k\} \ ,$$

and $[R_T(\delta)]^c = S_T-R_T(\delta)$ be the complement of $R_T(\delta)$ with respect to S_T, then, for sufficiently small δ,

$$\begin{aligned} & p \lim_{T\to\infty} [T^{-2}\sup_{\underset{\sim}{f}\in[R_T(\delta)]^c} \{\psi_X^{(T)}(\underset{\sim}{f})\}] \\ & = p \lim_{T\to\infty} [T^{-2} \ \Sigma_{j=1}^k(\alpha_{j,0}^2+\beta_{j,0}^2)\sup_{\underset{\sim}{f}\in[R_T(\delta)]^c}\{|H^{(T)}(f_j-f_{j,0})|^2\}] \\ & = (\Delta f)^2 \ \Sigma_{j=1}^k(\alpha_{j,0}^2+\beta_{j,0}^2)\{p \lim_{T\to\infty}[\sin^2(\pi\delta)/T^2\sin^2(\pi\delta/T)]\} \\ & < (\Delta f)^2 \ \Sigma_{j=1}^k(\alpha_{j,0}^2+\beta_{j,0}^2) \\ & = p \lim_{T\to\infty} [T^{-2}\psi_X^{(T)}(\underset{\sim}{f}_0)] \ . \end{aligned}$$

where $\underset{\sim}{f}_0 = (f_{1,0}, f_{2,0}, \ldots, f_{k,0})$. Hence

$$p \lim_{T\to\infty} [T(f_j^{(T)} - f_{j,0})] = 0 \ ,$$

and the theorem follows.

These results have been checked by simulating a number of series consisting of cosinusoids with discrete frequencies fairly close to each other, with varying relative amplitudes, and with and without an additional i.i.d. Gaussian error component having a standard deviation the size of the largest amplitude. We can report that in every case so far considered, all frequencies in each of the generated series were recovered with an error of about 1/10T by using the approach outlined in this paper. Specific details of these simulations will appear elsewhere.

For the case in which the error series $\varepsilon(t)$ is not an i.i.d. sequence of random variables, results have been obtained for periodogram estimates of frequency by Hannan (1971, 1973) and by Walker (1973). Hannan, in his 1971 paper, assumes $\varepsilon(t)$ to be a linear process; that is,

$$\varepsilon(t) = \Sigma_u \, g(u)\xi(t-u), \quad \Sigma_u |g(u)| < \infty \quad , \tag{2.29}$$

where the $\xi(t)$ form a pure noise series with zero mean and unit variance and possess a continuous spectral density. (Hannan refers to this as Condition A.) The error series in (2.29) is strictly stationary with finite second moments. Walker (1973) expands the form of (2.29) slightly by assuming that the $\xi(t)$ are also Gaussian with zero mean and finite variance, and that the coefficients are known functions, $g(u,\cdot)$, of an unknown parameter vector $\underset{\sim}{\theta}$. Hannan (1973) makes a more general assumption that $\varepsilon(t)$ is a strictly stationary ergodic process with mean zero and finite variance, and then obtains an almost sure convergence result for the periodogram estimator of a single (k=1) cosinusoidal component. Such a result may be extended to apply to the methods of this paper through the relationship (2.22) above.

3. FREQUENCY SEARCH PROCEDURES

The graph of the amplitude density function, $a_X^{(T)}(f)$, $0 < f < ½$, will typically display a fairly large number of local peaks, say, m^*. The next logical step, therefore, is to separate out the k significant peaks $(k \leq m^*)$ from those that more properly represent the noise level of the time series in question. Tests for identifying a dominant set of periodicities have traditionally been based on the largest few ordinates of the graph of the periodogram of the series; see Fisher (1929, 1940), Hartley (1949), Siegel (1980), and Section 6.1.4 of Priestley (1981). The problem is also related to the "bump-hunting" problem of Good and Gaskins (1980).

For series in which multiple frequencies are present, a stepwise search procedure is outlined on p. 34 of Walker (1971); there, it is suggested that one frequency at a time be estimated by examining the largest periodogram ordinate of the residual series after fitting all previously determined frequency components. Versions of Walker's suggestion have appeared in recent articles by Hill (1977) and by Damsleth and Spjøtvoll (1982). Hill, on the one hand, in what must have amounted to a veritable tour de force, claims to have used a computational technique involving over 100 iterations to estimate each of forty selected frequencies (one at a time) in the series of monthly sunspot numbers (the actual iteration and search techniques are not explained in the paper, however). On the other hand, Damsleth and Spjøtvoll, also using essentially Walker's stepwise algorithm, end up by selecting only eight frequencies in the annual sunspot number series. A search procedure for the sunspot series which results in a reasonable compromise between two such extremes, therefore, appears to be a worthwhile goal.

In Siddiqui and Izenman (1981), a two-stage search procedure is described to estimate k and determine those frequencies $f_1^{(T)}, f_2^{(T)}, \ldots, f_k^{(T)}$ which are prominent in the series $X(t)$; the first stage consists of a straightforward adaptation of Fisher's well-known test for periodicities, while the second stage uses information about the error spectrum to make the final choice. Here, we review that method and compare it with an adaptation of Hartley's (1949) analysis of variance based method using a maximum-F ratio statistic for fitting one frequency at a time. We have found, empirically, for a number of series, that the latter method yields results that are very close to those obtained by the Siddiqui-Izenman procedure, yet possesses the attractive feature that it is semi-automatic and can be carried out using only a least-squares regression program. Simulation studies of both search procedures for a variety of situations are clearly needed for future guidance.

3.1. The Siddiqui-Izenman Procedure

Preliminary search. For the jth frequency, $f_j^{(T)}$, identified as one of the m^* local peaks in the amplitude density function, compute the following quantities:

$$a_j^{(T)} = \frac{2}{T} \Sigma_{t=1}^{T} (X(t)-\bar{X}) \cos(2\pi f_j^{(T)} t), \tag{3.1}$$

$$b_j^{(T)} = \frac{2}{T} \Sigma_{t=1}^{T} (X(t)-\bar{X}) \sin(2\pi f_j^{(T)} t), \tag{3.2}$$

and

$$[c_j^{(T)}]^2 = [a_j^{(T)}]^2 + [b_j^{(T)}]^2, \tag{3.3}$$

where $\bar{X} = T^{-1}\Sigma_{t=1}^{T} X(t)$, for $j=1,2,\ldots,m^*$, and order the $f_j^{(T)}$, $j=1,2,\ldots,m^*$, according to the relative magnitudes of their $[c_j^{(T)}]^2$, $j=1,2,\ldots,m^*$. Thus, if

$$[c_{(1)}^{(T)}]^2 \geq [c_{(2)}^{(T)}]^2 \geq \cdots \geq [c_{(m^*)}^{(T)}]^2 \tag{3.4}$$

are the ordered values of (3.3), this, in turn, orders the frequencies in a corresponding fashion as $f_{(1)}^{(T)}, f_{(2)}^{(T)},\ldots, f_{(m^*)}^{(T)}$. Next, apply Fisher's test for periodicity to these m^* ordered frequencies. This test (Fisher 1929) is designed to test H_0: $k=0$ against H_1: $k \geq 1$, where k is the true number of discrete frequencies in the process under review. Although the test requires $[c_r^{(T)}]^2$ to be computed for each of the N Fourier frequencies $f_r = r/T$, $r=1,2,\ldots,N$, there exists an integer r_j, between 1 and N, such that

$$|f_j^{(T)} - \frac{r_j}{T}| \leq \frac{1}{2T}, \tag{3.5}$$

and so, for large T, the value of $[c_j^{(T)}]^2$ in (3.3) should be fairly close to the value of $[c_{r_j}^{(T)}]^2$ corresponding to r_j/T. Thus, the generalization of Fisher's test (Stevens 1939; Fisher 1940) will be approximately valid for each of the m^* normalized versions of (3.4)

$$g_j^{(T)} = [c_{(j)}^{(T)}]^2/2s_T^2, \quad j=1,2,\ldots,m^*, \tag{3.6}$$

where s_T^2 is the total variance of the observed series. Set $\alpha = 0.05$ or 0.01, and define the percentile $g_{j,T,\alpha}$ by the relation

$$\mathrm{Prob}(g_j^{(T)} \geq g_{j,T,\alpha}) = \alpha . \tag{3.7}$$

Values of $g_{j,T,\alpha}$ have been tabulated by Shimshoni (1971) for α=0.01 and 0.05, and for (i) $N = 5(5)50$, j=1,2,5,7,10, and (ii) $N = 100(100)3000$, j=1,2,5,10,25,50. Now, compare the value of $[c_{(j)}^{(T)}]^2$ with the appropriate value of $2s_T^2 g_{j,T,\alpha}$, and carry out this comparison according, to the following recipe suggested by Shimshoni. First, take the value of $g_{1,T,\alpha}$ and pass all frequencies whose $[c^{(T)}]^2$ values are greater than $2s_T^2 g_{1,T,\alpha}$. Let the j_1^{st} frequency be the first one to fail this test ($j_1 > 1$). Next, take the value of $g_{j_1,T,\alpha}$ and pass all further frequencies whose $[c^{(T)}]^2$ values are greater than $2s_T^2 g_{j_1,T,\alpha}$. When a frequency fails this second test, say, the j_2^{th} frequency, take the value of $g_{j_2,T,\alpha}$ and pass all frequencies for which their $[c^{(T)}]^2$ values are greater than $2s_T^2 g_{j_2,T,\alpha}$. And so on. The process ends when a new value of $2s_T^2 g$ is greater than the value of $[c^{(T)}]^2$ it was supposed to test. All remaining frequencies are then failed at this level of significance. The number of frequencies that are retained (out of the original m^*) is denoted by m_1. In certain situations, major frequencies may be hidden by virtue of their close proximity to more powerful frequencies. The next step is to screen the residual series

$$\epsilon^{(T)}(t) = X(t) - \mu_1^{(T)}(t), \quad t=1,2,\ldots,T, \tag{3.8}$$

where $\mu_1^{(T)}(t)$ is the estimated regression function consisting of the above m_1 frequency components; the $2m_1+1$ coefficients, α_0 and (α_j, β_j), $j=1,2,\ldots,m_1$, in the regression function

$$\mu_1(t) = \alpha_0 + \Sigma_{j=1}^{m_1}\{\alpha_j\cos(2\pi f_{(j)}^{(T)}t) + \beta_j\sin(2\pi f_{(j)}^{(T)}t)\} \tag{3.9}$$

are estimated by least-squares, using any standard regression computer program. Screening is done by computing the amplitude density function of (3.8), and repeating all the above tests. If m_2 denotes the number of frequencies declared significant at this second-step, then $m=m_1+m_2$ is the total number of significant frequencies detected.

Final Search. Estimate the $2m+1$ coefficients, α_0 and (α_j, β_j), $j=1,2,\ldots,m$, in the regression function

$$\mu(t) = \alpha_0 + \Sigma_{j=1}^{m}\{\alpha_j\cos(2\pi f_{(j)}^{(T)}t) + \beta_j\sin(2\pi f_{(j)}^{(T)}t)\},\ t=1,2,\ldots,T, \tag{3.10}$$

by least-squares, denote the coefficient estimates by $\alpha_0^{(T)}$ and $(\alpha_j^{(T)}, \beta_j^{(T)})$, $j=1,2,\ldots,m$, respectively, and let

$$|\gamma_j^{(T)}| = \{[\alpha_j^{(T)}]^2 + [\beta_j^{(T)}]^2\}^{\frac{1}{2}},\ \ j=1,2,\ldots,m. \tag{3.11}$$

Then, asymptotically, for large T (Grenander and Rosenblatt 1957, p. 246), the approximate covariances of (3.11) are

$$\mathrm{cov}\{|\gamma_i^{(T)}|, |\gamma_j^{(T)}|\} = \delta\{i-j\}g_\varepsilon(f_{(j)})/T,\ \ i,j=1,2,\ldots,m, \tag{3.12}$$

where $\delta\{\alpha\}$ is the Kronecker delta. If $g_\varepsilon^{(T)}(f)$ is a suitable estimator of $g_\varepsilon(f)$, and if $g_\varepsilon^{(T)}(f) \leq g_\varepsilon^{(T)}(f_*)$ for some f_*, then the bound

$$\hat{\text{var}}\{|\gamma_j^{(T)}|\} < g_\varepsilon^{(T)}(f_*)/T. \tag{3.13}$$

yields an approximate, but conservative, 95 percent confidence interval on $|\gamma_j| = \{\alpha_j^2 + \beta_j^2\}^{1/2}$, namely,

$$|\gamma_j^{(T)}| \pm 2\{g_\varepsilon^{(T)}(f_*)/T\}^{1/2}. \tag{3.14}$$

As long as any of the m frequencies has an associated confidence interval (3.14) which does not contain the value zero, then that frequency is declared significant and is retained in the final model. Otherwise, the frequency is dropped from further consideration. The value k is then taken to be the number of significant frequencies from this final selection stage.

3.2. An F-Directed Search Procedure

Another significance test for periodicities was proposed by Hartley (1949), who used a least-squares regression approach to finding a different normalization for the $[c_{(j)}^{(T)}]^2$ values in (3.4) than was used in (3.6). Since Hartley's procedure is not as well known as that of Fisher, we shall first review its salient points for the periodogram case.

From the N Fourier frequencies available, nominate m of them to be potentially significant. Denote these by r_j/T, $j = 1,2,\ldots,m$, where r_j is an integer between 1 and N. Next, set

$$X(t) = \alpha_0 + \Sigma_{j=1}^{m}\{\alpha_j \cos(2\pi r_j t/T) + \beta_j \sin(2\pi r_j t/T)\} + \varepsilon(t) , \tag{3.15}$$

where $\varepsilon(t)$ is Gaussian and satisfies assumption (A) above, and estimate (α_j, β_j), $j=1,2,\ldots,m$, by least-squares. The least-squares estimates are given by $(a_j^{(T)}, b_j^{(T)})$, $j=1,2,\ldots,m$, respectively (see (3.1), (3.2), where $f_j^{(T)} = r_j/T$, $j=1,2,\ldots,m$), and the fitted regression function by

$$X^{(T)}(t) = \overline{X} + \Sigma_{j=1}^{m}\{a_j^{(T)}\cos(2\pi r_j t/T) + b_j^{(T)}\sin(2\pi r_j t/T)\}. \qquad (3.16)$$

By virtue of the orthogonality of the trigonometric functions in (3.16), the regression sum of squares (SS_{reg}) can be expressed as the sum of m uncorrelated sums of squares; that is,

$$SS_{reg} = \Sigma_{t=1}^{T} (X^{(T)}(t)-\overline{X})^2 = \frac{T}{2}\Sigma_{j=1}^{m} [c_j^{(T)}]^2 \qquad (3.17)$$

(see (3.3)), where, under the hypothesis that $m = 0$ in (3.15), each $\frac{T}{2}[c_j^{(T)}]^2$ in (3.17) has an independent chi-squared distribution with two degrees of freedom (d.f.). The residual sum of squares (RSS_m) from the m-component regression (3.14), namely,

$$RSS_m = \Sigma_{t=1}^{T} (X(t)-X^{(T)}(t))^2 , \qquad (3.18)$$

has an independent chi-squared distribution with $T-2m-1$ d.f. Hartley's test statistic for individual (Fourier) frequencies is, then

$$F_{(j)} = \{\frac{T}{2}[c_{(j)}^{(T)}]^2/RSS_m\}(T-2m-1)/2 , \qquad (3.19)$$

for $j=1,2,\ldots,m$, where the $[c_{(1)}^{(T)}]^2,\ldots,[c_{(m)}^{(T)}]^2$ are the m largest values in (3.4) (cf. (3.6)). As an approximation to the upper percentage points of the distribution of (3.19), Hartley suggested that the (Fourier) frequency corresponding to $[c_{(j)}^{(T)}]^2$ be declared significant at the α% level if $F_{(j)}$ is larger than the tabulated $(\alpha/(m-j+1))$% point of the F distribution with 2 and $T-2m-1$ d.f., $j=1,2,\ldots,m$. The value of k is then taken to be that number of (Fourier) frequencies that are declared significant by this method.

Typically, however, the m largest local peaks in the graph of the amplitude density function will not be located at Fourier frequencies, and, consequently, the $[c_j^{(T)}]^2$, $j=1,2,\ldots,m$, will not be uncorrelated. We could, of course, argue, as we did in Section 3.1, that the differences between the frequency estimates and their nearest Fourier frequencies will be relatively small (see (3.5)), so that, for $i\neq j$, $[c_i^{(T)}]^2$ and $[c_j^{(T)}]^2$ will be approximately uncorrelated, and hence that Hartley's technique will be approximately valid. However, we prefer instead — both for computational and for statistical reasons — to re-interpret Hartley's procedure in a slightly different way, and then use that particular formulation here.

It is not difficult to see that for the special case of Fourier frequencies Hartley's method is equivalent to the following multistage procedure, and, by virtue of the argument noted in the previous paragraph, is approximately equivalent for the more general case. Since at each stage we carry out a series of one-component regressions, consideration of a fairly large number of frequencies is computationally feasible, especially if T is large. Furthermore, because at each stage we select a frequency using an F statistic formulation, the overall procedure is called "F-directed" and resembles a "forward stepwise" regression algorithm.

In the following discussion we attend to the general case and write $f_j^{(T)}$, $j=1,2,\ldots,m$, for the m nominated frequencies; if these frequencies are Fourier frequencies, then $f_j^{(T)} = r_j/T$, $j=1,2,\ldots,m$, for some integer r_j between 1 and N.

Consider, then, the problem of selecting k frequencies for the regression model

$$X(t) = \alpha_0 + \Sigma_{j=1}^{k}\{\alpha_j \cos(2\pi f_{(j)}^{(T)}t) + \beta_j \sin(2\pi f_{(j)}^{(T)}t)\} + \varepsilon(t) \ , \qquad (3.20)$$

where $f_{(1)}^{(T)}$ is the first frequency selected, $f_{(2)}^{(T)}$ is the second frequency selected, and so on, from a set of m potentially significant frequencies $f_j^{(T)}$, $j=1,2,\ldots,m$, $k \leq m$, and where $\varepsilon(t)$ is Gaussian and satisfies assumption (A). The selection rule is the following.

1. At step 1, let $F_{max}^{(1)}$ be the largest of the m F-ratio statistics, $F_j^{(1)}$, $j=1,2,\ldots,m$, obtained as follows. Regress $X(t)$ on the pair of components $(\cos(2\pi f_j^{(T)}t), \sin(2\pi f_j^{(T)}t))$, and obtain the regression sum of squares, $SS_{reg,j}^{(1)}$, having 2 d.f. Then, compute the statistic

$$F_j^{(1)} = \{SS_{reg,j}^{(1)}/RSS_m\}(T-2m-1)/2 \ , \qquad (3.21)$$

where RSS_m is the residual sum of squares obtained by regressing $X(t)$ on all m components $(\cos(2\pi f_j^{(T)} t), \sin(2\pi f_j^{(T)}t), j=1,2,\ldots,m)$. Clearly, $F_j^{(1)}$ has the F-distribution with 2 and T-2m-1 d.f. Set

$$F_{max}^{(1)} = \max\{F_j^{(1)}, j=1,2,\ldots,m\} \ . \qquad (3.22)$$

If $F_{max}^{(1)}$ is "significant", then declare the frequency corresponding to $F_{max}^{(1)}$ as being significant, and denote it by $f_{(1)}^{(T)}$. Compute the residuals from $X(t)$ after fitting the regression function (3.20) with $k=1$.

2. At step 2, let $F_{max}^{(2)}$ be the largest of the $m-1$ F-ratio statistics, $F_j^{(2)}$, $j=1,2,\ldots,m$, $j\neq(1)$, obtained as follows. Regress the residuals from step 1 on the pair of components $(\cos(2\pi f_j^{(T)}t), \sin(2\pi f_j^{(T)}t))$, and obtain the regression sum of squares, $SS_{reg,j}^{(2)}$, having 2 d.f. Then, compute the statistic

$$F_j^{(2)} = \{SS_{reg,j}^{(2)}/RSS_{m-1}\}(T-2m+1)/2 \quad . \tag{3.23}$$

where RSS_{m-1} is the residual sum of squares obtained by regressing the residuals from step 1 on all the remaining $m-1$ components $(\cos(2\pi f_j^{(T)}t), \sin(2\pi f_j^{(T)}t), j=1,2,\ldots,m, j\neq(1))$. As before, $F_j^{(2)}$ has an F-distribution, but with 2 and $(T-2m+1)$ d.f. Set

$$F_{max}^{(2)} = \max\{F_j^{(2)}, j=1,2,\ldots,m, j\neq(1)\} \quad . \tag{3.24}$$

If $F_{max}^{(2)}$ is "significant", then declare the frequency corresponding to $F_{max}^{(2)}$ as being significant, and denote it by $f_{(2)}^{(T)}$. Compute the residuals from $X(t)$ after fitting the regression function (3.20) with $k=2$.

3. In general, let $f_{(1)}^{(T)}, f_{(2)}^{(T)}, \ldots, f_{(r)}^{(T)}$ be the r frequencies declared significant at the conclusion of step r. At step $r+1$, let $F_{max}^{(r+1)}$ be the largest of the $m-r$ F-ratio statistics, $F_j^{(r+1)}$, $j=1,2,\ldots,m$, $j\neq(1),(2),\ldots,(r)$, as follows. Regress the residuals from step r on

the pair of components $(\cos(2\pi f_j^{(T)}t), \sin(2\pi f_j^{(T)}t))$, and obtain the regression sum of squares, $SS_{reg,j}^{(r+1)}$, having 2 d.f. Then, compute the statistic

$$F_j^{(r+1)} = \{SS_{reg,j}^{(r+1)}/RSS_{m-r}\}(T-2m+2r-1)/2 , \qquad (3.25)$$

where RSS_{m-r} is the residual sum of squares obtained by regressing the residuals from step r on all the remaining $m-r$ components $(\cos(2\pi f_j^{(T)}t), \sin(2\pi f_j^{(T)}t), j=1,2,\ldots,m, j\neq(1),(2),\ldots,(r))$, and set

$$F_{max}^{(r+1)} = \max\{F_j^{(r+1)}, j=1,2,\ldots,m, j\neq(1),(2),\ldots,(r)\}. \qquad (3.26)$$

If $F_{max}^{(r+1)}$ is "significant", then declare the frequency corresponding to $F_{max}^{(r+1)}$ as being significant, and denote it by $f_{(r+1)}^{(T)}$. If $F_{max}^{(r+1)}$ is "not significant", then terminate the selection procedure. The value of k is then taken to be the number of frequencies decleared to be significant according to the scheme.

The question of "significance" in the above algorithm relates to the problem of comparing, at any particular step, the observed value of F_{max} to the tabulated upper percentage points of the distribution of the largest order statistic from a set of correlated F variables. Since, at step r $(1 \leq r \leq m)$, the denominators of each of the $\{F_j^{(r)}\}$ are identical, results obtained for the multivariate F distribution (Gupta and Sobel 1962; Krishnaiah and Armitage 1965) would appear to be applicable.

However, the numerators of the $\{F_j^{(r)}\}$ are a correlated set of χ_2^2 variables, which, in turn, leads to more complicated results for the distribution theory. The paper by Krishnaiah and Rao (1962) may be useful here in defining a "generalized" multivariate F distribution; see Johnson and Kotz (1972, Section 40.8). For the present paper, however, we shall use a Bonferroni-type approximation to the upper percentage points of F_{max} similar to that suggested by Hartley for the independent case. Thus, at step r $(1 \leq r \leq m)$, declare $F_{max}^{(r)}$ to be significant at the α% level if $F_{max}^{(r)}$ is larger than the tabulated $(\alpha/(m-r+1))$% point of the F distribution with 2 and $T-2(m-r+1)-1$ d.f. (We note here that alternative approximations may be obtained from the multivariate probability inequalities of Karlin and Rinott (1980).)

If additional significant frequencies are suspected, but which may be hidden behind the more powerful ones, the following procedure will be appropriate. First, test as above for those m_1 frequencies which are considered potentially significant. Let k_1 of them be declared significant. Then, obtain the residual series by fitting all k_1 frequencies to $X(t)$, and recompute the amplitude density function for the residual series. If any additional frequencies appear to have relatively large ordinates in the graph of that function, add these, say m_2 in number, to the k_1 previously tested, to arrive at a total of $m = k_1 + m_2$. Then, repeat the above F-directed search procedure for all m nominated frequencies.

4. ANALYSIS OF THE ANNUAL SUNSPOT NUMBERS

The techniques of this paper will now be illustrated using the well-known time series of annual sunspot relative numbers from 1749 to 1979, a period of $T = 231$ years. The data are taken from Waldmeier (1961, pp. 20-21) for the years 1749-1960, and from Izenman (1981, Table 1.1) for the remaining years.

The sunspot numbers have received a (well-deserved) reputation in the statistical literature for being one of the most difficult series to model successfully, primarily because of its nonstandard features (Bloomfield 1976). We emphasize here that the analysis presented in this Section is not claimed to be the best for predicting sunspot behavior in the future; indeed, the nonGaussian, nonstationary, aperiodic, and non-linear behavior of the series make its fitting and prediction an extremely complex affair. However, it is precisely because of these nonstandard features that make the sunspot numbers such an interesting and nontrivial example for demonstrating the methodology of this paper.

4.1. The Annual Sunspot Numbers

The graph of the amplitude density function (2.20) with $\Delta f = 10^{-4}$ of the mean-corrected annual sunspot numbers is displayed in Figure 1. The prominent features of that graph include two clusters of large peaks, one at the very low frequencies, and the other at frequencies around the so-called "11-year sunspot period". Ninety-eight local maxima were located within the frequency band $0 \leq f \leq \frac{1}{2}$. The next step, therefore, is to screen these peaks for significant frequencies. We shall use the two search procedures described in Section 3.

(1) First, we consider the search method outlined in Section 3.1. The total variance of the series $s_T^2 = 1583.32826$. If we carry out the procedure as described, we pass a total of nine frequencies, a number comparable to that obtained by Damsleth and Spjøtvoll (1982). However, when compared to results obtained by Hill (1977), Rust and Kirk (1977), and Siddiqui and Izenman (1981) for the monthly sunspot numbers, and since it can be argued that the major frequencies in the annual series should be similar in number and location to those major low frequencies in the monthly series, we conclude that, for some reason, too few frequencies are being passed here. The problem lies in the fact that the annual sunspot numbers are computed from the monthly numbers by averaging in disjoint blocks of length twelve, thereby smoothing out the highest amplitudes of the monthly sunspot numbers. An adjustment is necessary, therefore — in a downwards direction — to the entries in Shimshoni's Tables to take this into account. We propose to do this here by multiplying Shimshoni's critical values, $g_{j,T,\alpha}$, by a "compensation factor", h_{12}^2, say, $0 \leq h_{12} \leq 1$, which we take to be a location parameter of the distribution of the ratio of the mean to the maximum in a sample of size twelve from some appropriate underlying distribution defined on $[0,\infty)$. In general, this problem is nontrivial and remains open. A related reference is the paper by Maller and Resnick (1981), which considers the limiting behavior for large sample sizes of the above ratio statistic; however, their results do not seem to be of help to us.

Since the results we need are not currently available, we are led to the following approximation. From the monthly sunspot numbers, we computed the 231 ratios of annual mean to "annual maximum" (that is, the maximum monthly sunspot number for that year), and then set $h_{12} = 0.62$, the median of all those values. The histogram of the annual ratios is displayed in Figure 2.

Thus, $2s_T^2h_{12}^2g_{1,100,0.05} = 89.81$. This critical value passes the following eight frequencies:

j	$f_{(j)}^{(T)}$	$p_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
1	.0902	11.087	780.7
2	.1005	9.950	616.3
3	.0107	93.346	343.7
4	.0945	10.582	341.3
5	.1062	9.416	246.7
6	.0172	58.140	173.0
7	.0835	11.976	159.5
8	.0039	256.410	118.7

where $p_{(j)}^{(T)}$ is the period (in years) corresponding to $f_{(j)}^{(T)}$; that is, $p_{(j)}^{(T)} = 1/f_{(j)}^{(T)}$. Now, $j_1=9$, and, by interpolation in Shimshoni's Table, we get that $g_{9,100,0.05} = 0.02902$, whence $2s_T^2h_{12}^2g_{9,100,0.05} = 35.32$. This second critical value passes eight further frequencies, namely,

j	$f_{(j)}^{(T)}$	$p_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
9	.1168	8.562	76.3
10	.0772	12.953	71.7
11	.0232	4.698	66.1
12	.1115	8.969	54.7
13	.0716	13.966	54.5
14	.1233	8.110	48.8
15	.0473	21.142	46.6
16	.0668	14.970	43.0

At the next stage, $j_2=17$, so that $g_{17,100,0.05} = 0.02177$, and $2s_T^2h_{12}^2g_{17,100,0.05} = 26.50$. The next largest value of $[c^{(T)}]^2$ is xx.x, corresponding to the frequency .xxxx; however, this value is too low to be passed by this test.

The value of m_1 is, therefore, 16. A computation of the resulting sixteen-component regression function, and of the amplitude density function of the residual series, revealed an additional frequency which is passed by the above test, namely,

j	$f_{(j)}^{(T)}$	$p_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
17	.1199	8.340	46.1

so that $m_2=1$, and hence $m=17$.

Now, we make a final selection from these seventeen frequencies. After computing the fitted regression function using these seventeen frequency components, we examined the autocorrelation and partial autocorrelation functions of the residual series, and determined that the best, and most parsimonious fit to that latter series would be a second-order autoregressive process of the form

$$\varepsilon^{(T)}(t) = \phi_1\varepsilon^{(T)}(t-1) + \phi_2\varepsilon^{(T)}(t-2) + \delta(t),$$

where $\delta(t)$ is a white noise process with mean zero and variance σ_δ^2. The maximum-likelihood estimates and associated standard errors of the two parameters in the model are:

parameter	MLE	SE
ϕ_1	.6736	.0652
ϕ_2	-.3722	.0662

and $\hat{\sigma}_\delta^2 = 146.4$. The roots of the corresponding characteristic equation are both complex, indicating pseudo-periodic behavior of the $\varepsilon^{(T)}(t)$-process (Box and Jenkins 1976, pp. 58-63). The estimated spectral density function of the residual series is, therefore, given by

$$g_\varepsilon^{(T)}(f) = 2\hat{\sigma}_\delta^2 / \{(1+\hat{\phi}_1^2+\hat{\phi}_2^2)-2\hat{\phi}_1(1-\hat{\phi}_2)\cos(2\pi f)-2\hat{\phi}_2\cos(4\pi f)\}$$

$$= 292.8/\{1.5923-1.8486\cos(2\pi f)+0.7444\cos(4\pi f)\},$$

where $0 \leq f \leq \frac{1}{2}$. The maximum of $g_\varepsilon^{(T)}(f)$ occurs at $f = f_*$, where

$$\cos(2\pi f_*) = \hat{\phi}_1(1-\hat{\phi}_2)/(-4\hat{\phi}_2) = 0.6208;$$

that is, $f_* = 0.1434$. Thus, $g_\varepsilon^{(T)}(f) \leq g_\varepsilon^{(T)}(f_*) = 1068.2325$, so that $\hat{\text{var}}\{|\gamma_j^{(T)}|\} < 1068.2325/231 = 4.6244$. Conservative 95 percent confidence limits on $|\gamma_j|$ are given by $|\gamma_j^{(T)}| \pm 4.3009$. Examination of the seventeen values of $|\gamma_j^{(T)}|$ showed that these limits passed fourteen out of the seventeen frequencies passed at the preliminary screening; the three frequencies that were eliminated are .1168, .0716, and .1115. Table 1 shows the final fitted regression coefficient estimates for the fourteen-component function, and Figure 3 gives a visual display of the observed series of annual sunspot numbers and the fitted regression function. The amount of total variance explained by the fitted function is 85.4 percent.

(2) Next, we consider the F-directed search method as described in Section 3.2. Eighteen frequencies were nominated as potentially significant, which were also the largest eighteen peaks in the graph of the amplitude density function (see Figure 1). The steps in the search procedure are set out in Table 2, where we have taken $\alpha = 1.0$ and $m = 18$. At step r, we compare the value of $F_{max}^{(r)}$ to the $(1/(19-r))\%$ point of

the F distribution with 2 and $192+2r$ d.f., and note that

$$F_{2,\infty}^{0.1\%} = 6.91\ ,\quad F_{2,\infty}^{0.5\%} = 5.30,\quad F_{2,\infty}^{1.0\%} = 4.61\ . \tag{4.1}$$

Thus, 15 out of the 18 nominated frequencies were found significant by this method, and the amount of total variance explained by the fitted function is 86.4 percent.

To summarize the above results, we note the following. First, we were able to separate the two major frequencies .0902 (11.087 years) and .0945 (10.582 years) from each other which are less than 1/T apart. Second, although periodogram analysis of the annual sunspot numbers exhibits a "broad peak of indeterminate width" extending from the frequency .0625 to the frequency .1250 (Bloomfield 1976, p. 96), we have shown that there are actually nine quite distinct and significant frequencies inside that band. We also identified a new major frequency in the series, .0039 or 256.41 years, which should be taken with some caution since its length exceeds the length of the series itself. However, it does suggest that the resolution of this technique is high enough to enable it to identify cycles from series whose length is less than the period of that cycle.

4.2. The Signed Annual Sunspot Numbers

Since the sunspot numbers are, by definition, nonnegative, this introduces an implicit, but major, restriction onto the form of the fitting function. Any unconstrained model that attempts to follow closely the course of the sunspot numbers over time will, therefore, be estimated by a function that will not necessarily be nonnegative at each of its minima. Such a feature may be seen, for example, in Figure 3, where certain very low minima are fitted by negative numbers. For many similar series, little or nothing can be done to alleviate this problem without making the model correspondingly more complicated. However, we are fortunate in that a very natural and physically meaningful data transformation is available to us which eliminates this problem and, furthermore, dramatically improves the performance of the model.

In 1908, G. E. Hale discovered the existence of the solar magnetic cycle, in which alternate 11-year sunspot cycles are accompanied by a reversal in polarity of the sun's magnetic field (Bray and Loughhead 1964). Loosely speaking, cycles having 'positive' magnetic polarity are invariably succeeded by cycles having 'negative' polarity, and vice-versa. These results led Hale in 1925 to conclude that a "complete" solar cycle was actually of the order of 22 years, from the start of a positive cycle through to the start of the next positive cycle. This fact, now well-established in the scientific literature, led Bracewell (1953) in a short letter to the journal Nature to suggest that perhaps sunspot numbers corresponding to years of negative magnetic cycles be assigned a minus sign, while sunspot numbers associated with years of positive cycles be unchanged. The result of such a transformation is a reflection through the time axis of every other 11-year cycle of sunspot numbers. Thus, the series would no longer be constrained in the above sense. Now, minima would be represented by the zero-crossing points of the graph of the series through the time axis, negative numbers would possess a natural physical interpretation, and maxima would be given by the sequence of alternating positive and negative extrema of the transformed series. For our purposes, it is necessary to assume that the solar magnetic cycle was present for at least three hundred years prior to its discovery. It is surprizing to note that although Bracewell's suggestion appeared thirty years ago, we could find only three articles in which similar suggestions were put forward; these were in a brief reference to Bracewell's letter by Moran (1954), a comment (but not followed up) in Brillinger and Rosenblatt (1967), p. 215, and in a recent note in Nature by Hill (1977).

A technical remark on the use of this transformation is in order here. Since, in most instances, a minimum occurs midway through a year, the question is raised as to whether that year be assigned a positive or a negative value. One way of settling this is to examine the series of monthly sunspot numbers for the minimum year in question, determine a month of minimum by inspection (which, in some cases, may not be unique or even obvious), assign plus and minus signs as appropriate up to and then following the month of minimum, and finally, recompute a new annual mean for that year. The general effect of such a recomputed transition sunspot number is to smooth the transition from positive years to negative years and vice-versa. Fortunately, it appears that precise determination of the month of minimum is not crucial for a successful application of the methods of this paper, and anyway, in some cases, a definitive minimum month will be difficult to resolve by means other than by guesswork.

The graph of the amplitude density function (2.20) with $\Delta f = 10^{-4}$ of the mean-corrected signed annual sunspot numbers is displayed in Figure 4. Ninety-four local maxima were located within the frequency band $0 \leq f \leq \frac{1}{2}$, and the most prominent feature visible in the graph is that the two clusters of frequencies in Figure 1 have now been replaced by a single cluster approximately midway between them. The transformation appears to have removed most of the longer cycles in the original series.

(1) The total variance of the transformed series is $s_T^2 = 4101.5426$, and from Shimshoni's Table 2(b), we again have $g_{1,100,0.05} = 0.07378$. We now need to make a similar argument as for the original series in Section 4.1 and come up with a value for h_{12} the compensation factor in going from the series of signed monthly sunspot numbers to the signed annual numbers. To obtain the signed monthly numbers, we would have to go back to the daily sunspot numbers; rather than do this, we note the following. For the "positive" cycles, the ratio of signed annual mean to signed annual maximum for each year will be the same as those for the original series, and for the "negative" cycles, the ratio of signed annual mean to signed annual maximum will be equal to the ratio of (original) annual mean to (original) annual minimum, which, in turn, will be greater than or equal to the ratio of (original) annual mean to (original) annual maximum. Hence, from the results of Section 4.1, we can assume, for the signed series, $0.62 \leq h_{12} \leq 1$. To allow maximum flexibility here, we take $h_{12} = 0.62$. Thus, $2s_T^2 h_{12}^2 g_{1,100,0.05} = 232.65$, which passes the following five frequencies:

j	$f_{(j)}^{(T)}$	$P_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
1	.0449	22.272	5374.0
2	.0557	17.953	1545.6
3	.0505	19.802	983.7
4	.0611	16.367	491.9
5	.0388	25.775	298.4

Now, $j_1 = 6$, and, by interpolation, we get $g_{6,100,0.05} = 0.03504$, whence $2s_T^2 h_{12}^2 g_{6,100,0.05} = 110.49$. This critical value passes two further frequencies, namely,

j	$f_{(j)}^{(T)}$	$P_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
6	.0343	29.255	161.5
7	.0269	37.175	132.6

Next, $j_2=8$, so that $g_{8,100,0.05} = 0.03104$, and $2s_T^2h_{12}^2g_{8,100,0.05} = 97.88$. This critical value passes a single additional frequency, namely,

j	$f_{(j)}^{(T)}$	$P_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
8	.0802	12.469	102.4

This time, $j_3=9$, $g_{9,100,0.05} = 0.02902$, and $2s_T^2h_{12}^2g_{9,100,0.05} = 91.51$, which passes the two frequencies

j	$f_{(j)}^{(T)}$	$P_{(j)}^{(T)}$	$[c_{(j)}^{(T)}]^2$
9	.1454	6.878	95.4
10	.1379	7.252	94.6

Now, $j_4=11$, $g_{11,100,0.05} = 0.02627$, and $2s_T^2h_{12}^2g_{11,100,0.05} = 82.84$. The next largest value of $[c^{(T)}]^2$ is 79.4, which corresponds to the frequency .1508 (or, 6.63 years), but which is too small to pass the test. Hence, $m_1=10$, and a screening of the residual series from the ten-component fit gives $m_2=0$, so that $m=10$.

As in Section 4.1 we make a final selection from these ten frequencies. The best, and most parsimonious, fit to the residual series after fitting a ten-component regression function from the above frequencies is a second-order autoregressive process of the form (4.1), where the estimated parameters and their standard errors are:

parameter	MLE	SE
ϕ_1	.8945	.0640
ϕ_2	-.3523	.0641

and $\hat{\sigma}_\delta^2 = 183.1$. As before, the roots of the characteristic equation are both complex, indicating pseudo-periodic behavior, and the estimated spectral density function of the residual series is given by

$$g_\varepsilon^{(T)}(f) = 366.2/\{1.9242-2.4193\cos(2\pi f)+0.7046\cos(4\pi f)\},$$

where $0 \leq f \leq \frac{1}{2}$, with a maximum value at $f = f_* = (2\pi)^{-1}\cos^{-1}(0.8584) = 0.0857$. Hence, $g_\varepsilon^{(T)}(f) \leq g_\varepsilon^{(T)}(f_*) = 2019.38$, so that $\hat{\text{var}}\{|\gamma_j^{(T)}|\} < 2019.38/231 = 8.74$. Conservative 95 percent confidence limits on $|\gamma_j|$ are, therefore, given by $|\gamma_j^{(T)}| \pm 5.9133$. These limits passed all but one of the frequencies passed at the preliminary stage, the frequency being eliminated was .0343. Table 3 lists the coefficient estimates of the nine-component regression function, and Figure 5 gives a visual display of the signed annual sunspot numbers and the fitted regression function. The amount of total variance of the signed series explained by the fitted function is 91.0 percent.

(2) For the F-directed search procedure, fifteen frequencies were nominated as potentially significant, which were also the largest peaks in the graph of the amplitude density function (see Figure 4). The steps in the search procedure are set out in Table 4, where we have taken $\alpha = 1.0$ and $m = 15$. At step r, we compare the value of $F_{max}^{(r)}$ to the $(1/(16-r))\%$ point of the F distribution with 2 and $198+2r$ d.f. (see (4.1)). Thus, 12 out of the 15 nominated frequencies were found significant by this method, and the amount of total variance explained by the fitted function is 92.6 percent.

4.3. The Frequency Trace

One way of judging the extent of nonstationarity in a time series is to complex demodulate the series at a specific frequency (Bingham, Godfrey, and Tukey 1967). When this is done for the annual sunspot numbers at a frequency corresponding to an 11-year cycle (Bloomfield 1976, pp. 137-140), substantial variations over time in both amplitude and phase become immediately visible. The techniques presented in this paper can now be used to complement that approach so that we can actually quantify the movement (if any) of any frequency estimate over time. Such a direct measurement of "frequency drift" is made possible only through a high-resolution frequency analysis.

If a process is truly stationary over time with fixed periodicities, then by extending the length of the physical record for that process by concatenating additional observations to either end, the resulting frequency analysis should not be radically effected. In other words, we should not expect the overall shape of the graph of the amplitude density function to alter perceptively by a lengthening of the series record. This argument suggests two distinct ways of obtaining a sequence of estimates of each major frequency of the series. They are the following:

(1) Fix an initial reference time point (which could be, for example, the first observation), compute the graph of the amplitude density function for an initial series length of T_1 values (where $T_1 < T$) starting from that initial time point, and thereby obtain the estimate $f_j^{(T_1)}$ of f_j. Next, increase the record length to T_2 observations by adding T_2-T_1 more recent data points to the T_1 values

already analyzed ($T_2 < T$), and repeat all previous computations to get a second estimate $f_j^{(T_2)}$ of f_j. Thus, by successively increasing the amount of data available on the process by, say, n steady increments of $T_{i+1}-T_i$ observations, $i=1,2,\ldots,n$, and where, typically, $T_{n+1} = T$, we can obtain a sequence of "forward estimates" $f_j^{(T_i)}$, $i=1,2,\ldots,n+1$, of f_j.

(2) The alternative procedure is to fix a final reference time point (the latest observation, perhaps), compute the amplitude density function for a series record of length T_1 terminating at that final time point ($T_1 < T$), and obtain the estimate $f_j^{(T_1)}$ of f_j. Now, increase the record length to T_2 observations by adding T_2-T_1 more ancient observations to the front end of the series, and obtain a second estimate $f_j^{(T_2)}$ of f_j. In this way, by increasing in a systematic fashion the amount of data available on the process, we can obtain a sequence of "reverse estimates" $f_j^{(T_i)}$, $i=1,2,\ldots,n+1$, of f_j, where $T_i < T_{i+1}$ and $T_{n+1} = T$.

In certain situations, such as for the sunspot numbers series, both approaches may be of independent interest.

Thus, if $f_j^{(T_i)}$, $i=1,2,\ldots,n+1$, denote the n+1 estimates of f_j as determined by either of the two methods above, we now plot $f_j^{(T_i)}$ against T_i, $i=1,2,\ldots,n+1$. If the corresponding period $p_j = 1/f_j$ is stable over time, the resulting plot should reveal an approximate straight-line parallel to the horizontal (time) axis. We call such a plot the frequency trace of $f_j^{(T)}$, and, to distinguish between the two types of estimates, we shall refer to the trace as either a forward frequency trace or as a reverse frequency trace of $f_j^{(T)}$ according as the initial or the final reference time point is held fixed. There is no reason to expect these two types of frequency traces to be identical. Particular and persistent deviations from a straight-line plot will give a quantitative description of the direction and extent of any nonstationarity present in the series.

In Figures 6a, b, and c, we have plotted frequency traces for the triplet of close frequencies from Table 1 around the so-called "11-year sunspot period", namely, $f_1^{(T)} = .0902$, $f_2^{(T)} = .1005$, and $f_4^{(T)} = .0835$, with $\Delta f = 10^{-4}$ and $T_{i+1}-T_i = 2$, $i=1,2,\ldots,n$, for the annual sunspot numbers. These plots each indicate nonstationarity of the series. For example, if we use all 231 available sunspot numbers for 1749-1979, the most prominent frequency is estimated to be .0902 (11.087 years); however, the most recent, and certainly the most reliable, data (at least, since 1930) suggest that it might be wiser to estimate it (say, for prediction purposes) as .0952 (10.504 years). Apparently, the "11-year sunspot cycle" has been getting systematically shorter with time. Others (such as Currie 1980) have also noticed this phenomenon, but by using different methods. We have also plotted in Figures 7a, b, and c the frequency traces for the three longest periods detected in the sunspot series (see Table 1), namely, $f_3^{(T)} = .0107$, $f_6^{(T)} = .0172$, and $f_{11}^{(T)} = .0039$. The rapid shifts in the various frequency estimates over time may have implications for the historians' claims of variability in the quality of the actual sunspot numbers used to compile the more ancient data (see Eddy 1976; Izenman 1981).

5. ACKNOWLEDGEMENTS

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Table 1. Estimates of parameters for the 14-component fit to the series of annual mean sunspot relative numbers, 1749-1979. Frequency components are ordered by decreasing magnitudes of the $|\gamma_j^{(T)}|$.

j	$f_j^{(T)}$	$P_j^{(T)}$	$\alpha_j^{(T)}$	$\beta_j^{(T)}$	$\lvert\gamma_j^{(T)}\rvert$
0	constant		49.019		
1	.0902	11.087	25.115	15.181	29.347
2	.1005	9.950	14.894	12.115	19.199
3	.0107	93.346	6.062	15.177	16.343
4	.0835	11.976	-11.246	10.807	15.597
5	.0945	10.582	9.059	-10.688	14.011
6	.0172	58.140	- 7.889	- 7.246	10.712
7	.1233	8.110	3.870	- 9.957	10.682
8	.1062	9.416	- 0.634	9.448	9.469
9	.1199	8.340	1.434	- 8.911	9.026
10	.0473	21.142	5.759	- 5.117	7.704
11	.0039	256.410	4.927	- 5.171	7.143
12	.0772	12.953	- 2.261	6.008	6.419
13	.0232	4.698	4.828	- 3.610	6.028
14	.0668	14.970	- 0.296	- 5.034	5.043

Table 2. Summary of ANOVA selection of significant frequencies for the time series of annual mean sunspot relative numbers, 1749-1979. Frequencies were entered into the model if their corresponding F_{max} statistic was significant at the 1% level (see text, Section 3.2). The total sum of squares for the series is 364655 with 230 d.f., and m was taken as 18.

r	$f_{(r)}^{(T)}$	$SS_{reg}^{(r)}$	RSS_{m-r+1}	$F_{max}^{(r)}$	R^2
1	.0902	90439	45317	193.58	25.8%
2	.1005	59578	46632	125.20	21.7%
3	.0107	40481	47818	83.81	52.5%
4	.0835	31159	49322	63.17	61.4%
5	.0945	20040	49671	40.75	67.3%
6	.0172	14896	42239	35.98	71.6%
7	.1233	10228	46359	22.72	74.4%
8	.1062	9279	47045	20.51	77.1%
9	.1199	7895	46828	17.70	79.5%
10	.0473	6167	46607	14.02	81.2%
11	.0039	4254	46761	9.74	82.4%
12	.0772	4012	46411	9.33	83.6%
13	.0232	3774	46284	8.88	84.7%
14	.1833	3508	46015	8.38	85.7%
15	.0668	2693	45937	6.50	86.4%

Table 3. Estimates of parameters for the nine-component fit to the series of signed annual mean sunspot relative numbers, 1749-1979. Frequency components are ordered by decreasing magnitudes of the $|\gamma_j^{(T)}|$.

j	$f_j^{(T)}$	$P_j^{(T)}$	$\alpha_j^{(T)}$	$\beta_j^{(T)}$	$\lvert\gamma_j^{(T)}\rvert$
0	constant		- 1.324		
1	.0449	22.272	68.812	11.227	69.722
2	.0557	17.953	14.650	23.496	27.689
3	.0388	25.775	-16.952	2.297	17.107
4	.0505	19.802	15.209	1.347	15.269
5	.0611	16.367	- 9.392	7.896	12.270
6	.0802	12.469	8.973	6.045	10.819
7	.0269	37.175	3.771	9.607	10.321
8	.1454	6.878	9.154	1.792	9.328
9	.1379	7.252	- 8.067	3.156	8.662

Table 4. Summary of ANOVA selection of significant frequencies for the time series of signed annual mean sunspot relative numbers, 1749-1979. Frequencies were entered into the model if theire corresponding F_{max} statistic was significant at the 1% level (see, text, Section 3.2). The total sum of squares for the series is 943355 with 230 d.f., and m was taken as 15.

r	$f_{(r)}^{(T)}$	$SS_{reg}^{(r)}$	RSS_{m-r+i}	$F_{max}^{(r)}$	R^2
1	.0449	626448	61198	1023.64	66.4%
2	.0557	106372	70863	151.61	77.9%
3	.0388	35333	66386	54.29	81.8%
4	.0505	24925	64885	39.57	84.7%
5	.0611	13641	62575	23.43	86.2%
6	.0802	13799	61830	22.67	87.6%
7	.0269	11989	61755	20.58	88.9%
8	.1454	10783	61640	18.72	90.1%
9	.1379	8278	62154	14.38	91.0%
10	.1287	5842	61765	10.31	91.6%
11	.1031	4652	61713	8.29	92.1%
12	.1508	4566	61667	8.22	92.6%

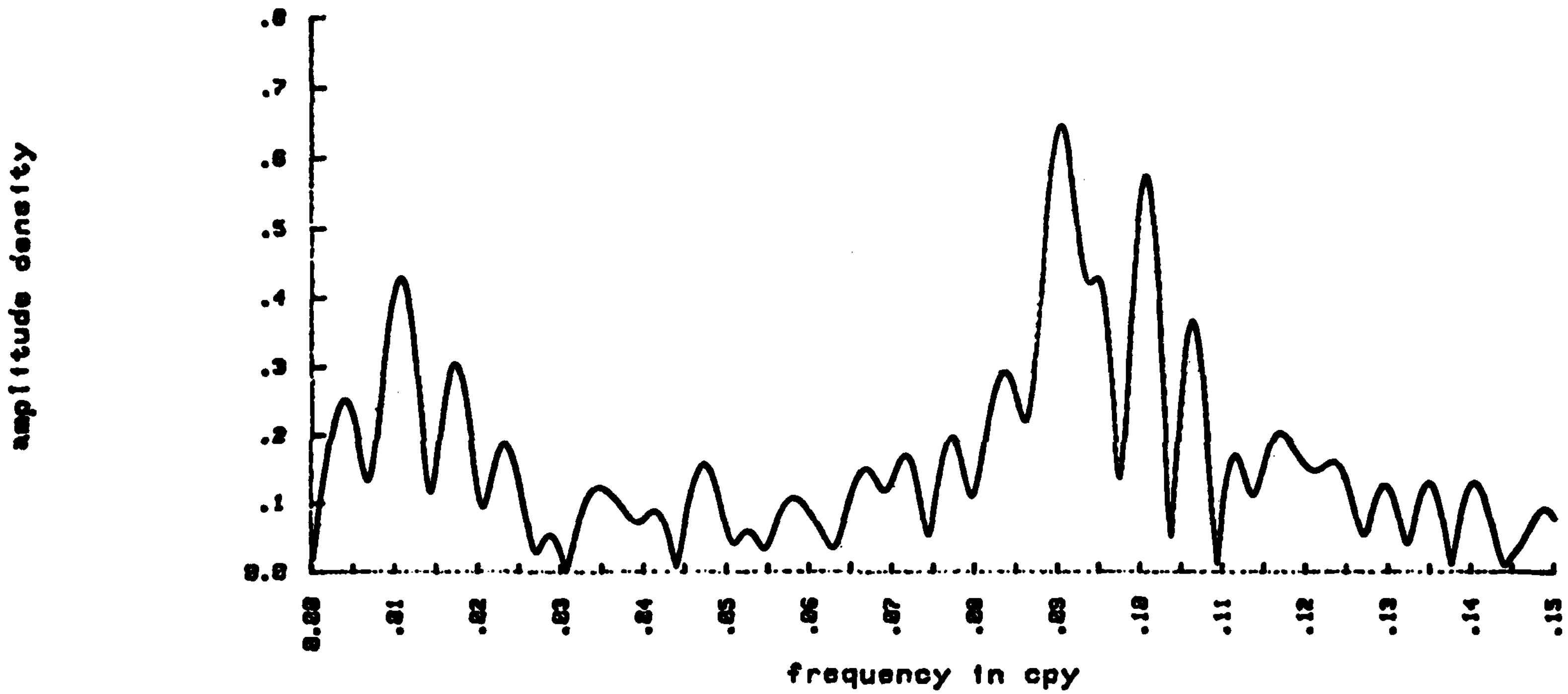

FIG. 1. Low frequency portion of the amplitude density function of annual sunspot relative numbers, 1749-1979, with $\Delta f = .0001$.

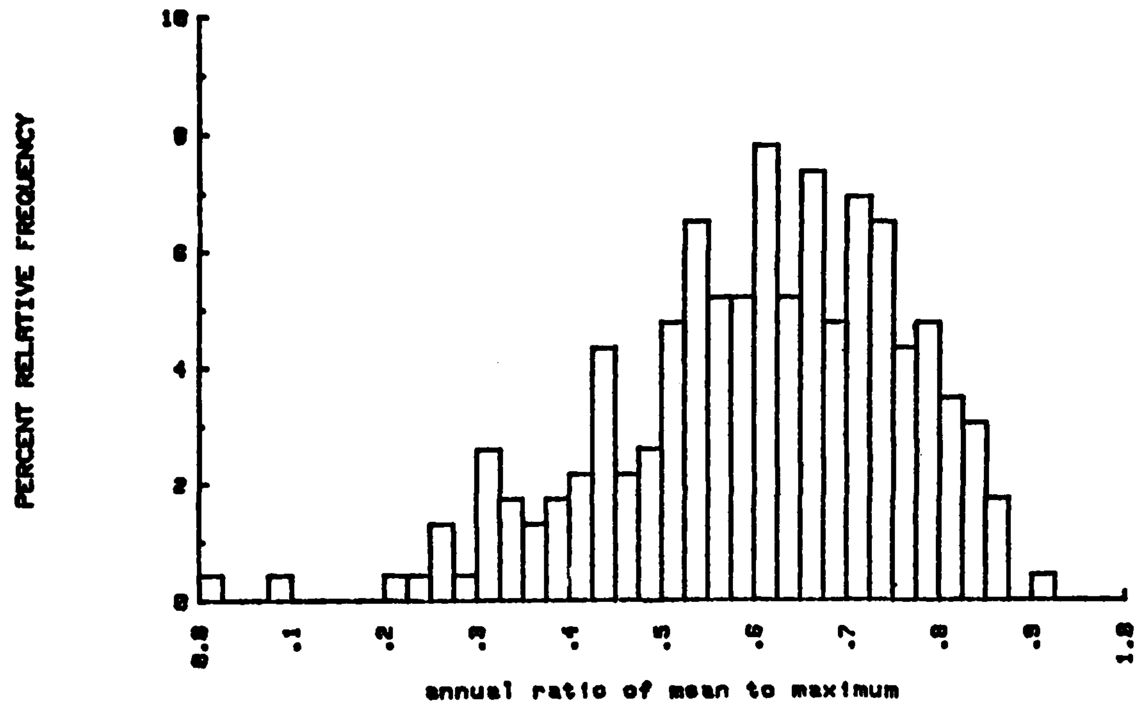

FIG. 2. Histogram of the 231 ratios of annual mean sunspot number to maximum monthly sunspot number that year for the period 1749-1979.

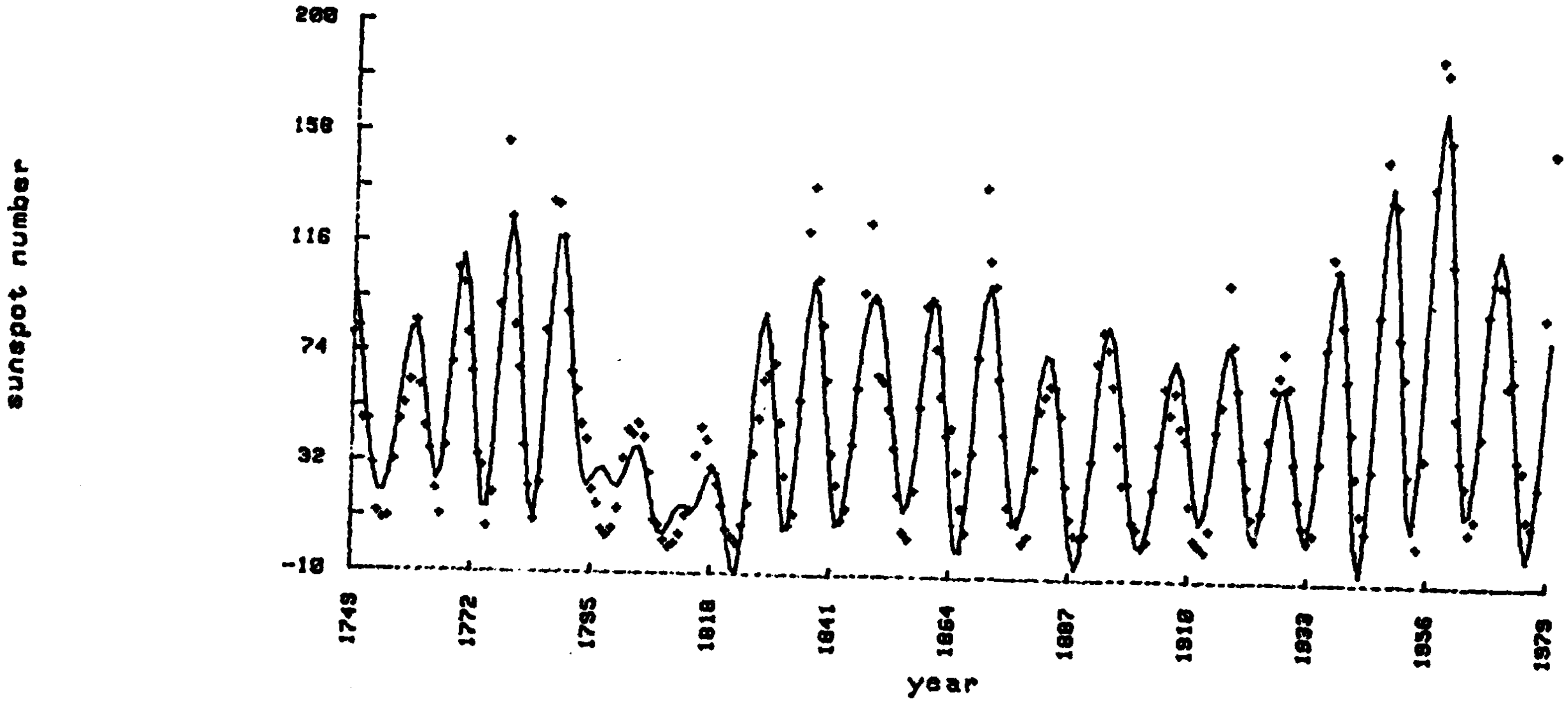

FIG. 3. Fourteen-component least-squares fit to annual sunspot relative numbers, 1749-1979. The symbol + indicates an observed value, the solid line is the fitted function.

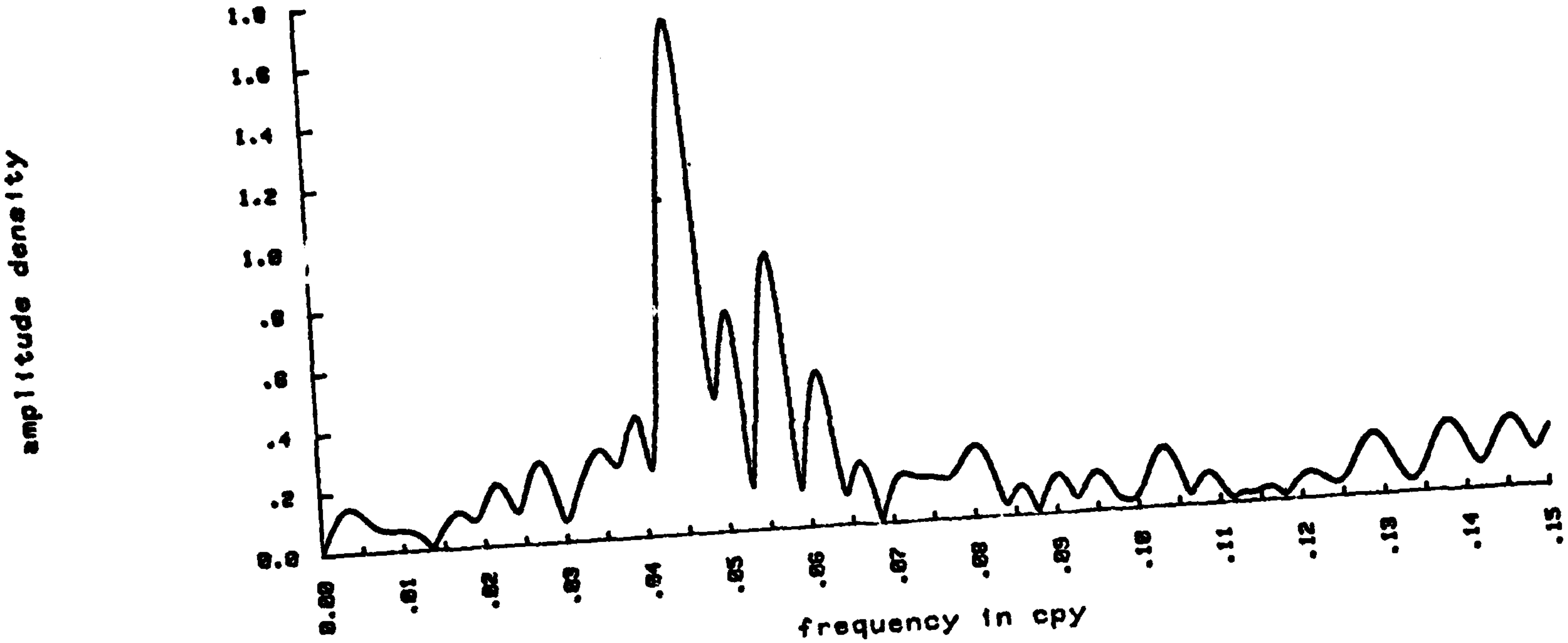

FIG. 4. Low frequency portion of the amplitude density function of signed annual sunspot relative numbers, 1749-1979, with $\Delta f = 0.0001$.

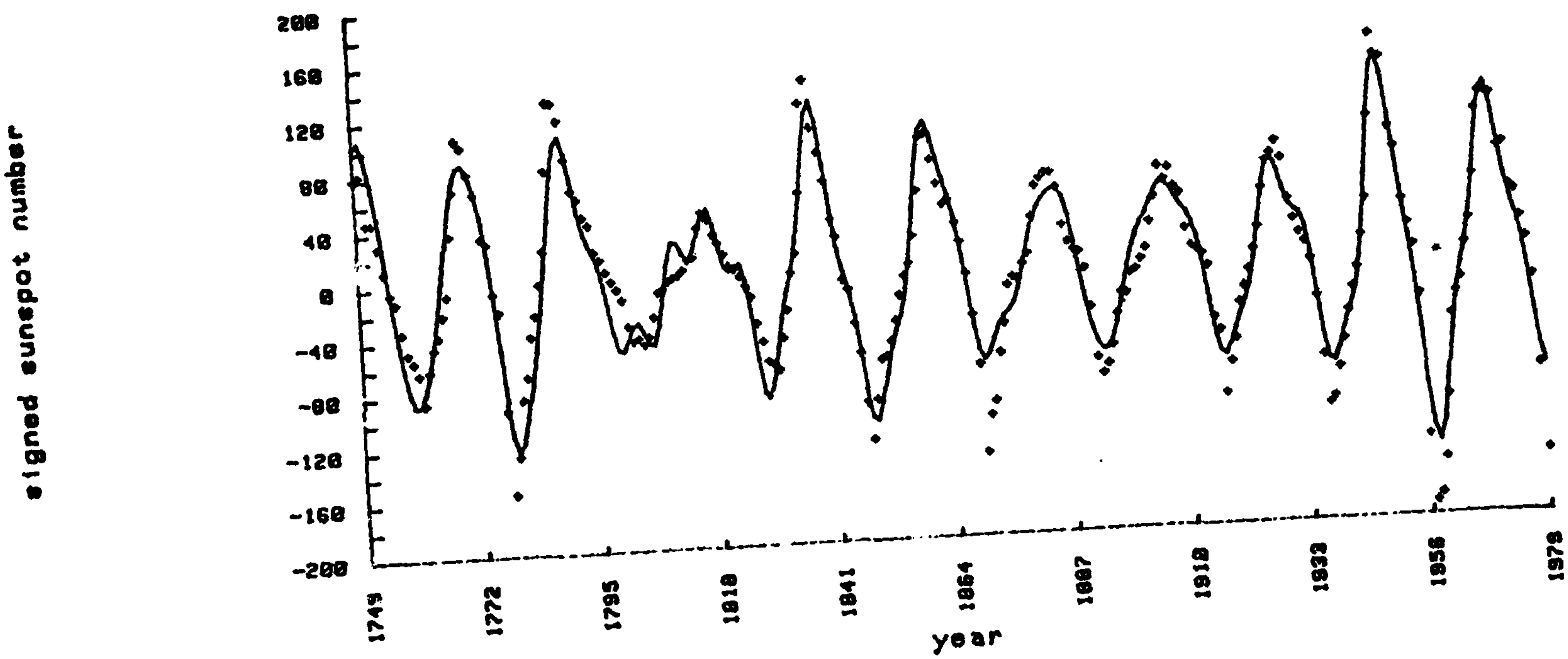

FIG. 5. Nine-component least-squares fit to signed annual sunspot relative numbers, 1749-1979. The symbol + indicates an observed value, the solid line is the fitted function.

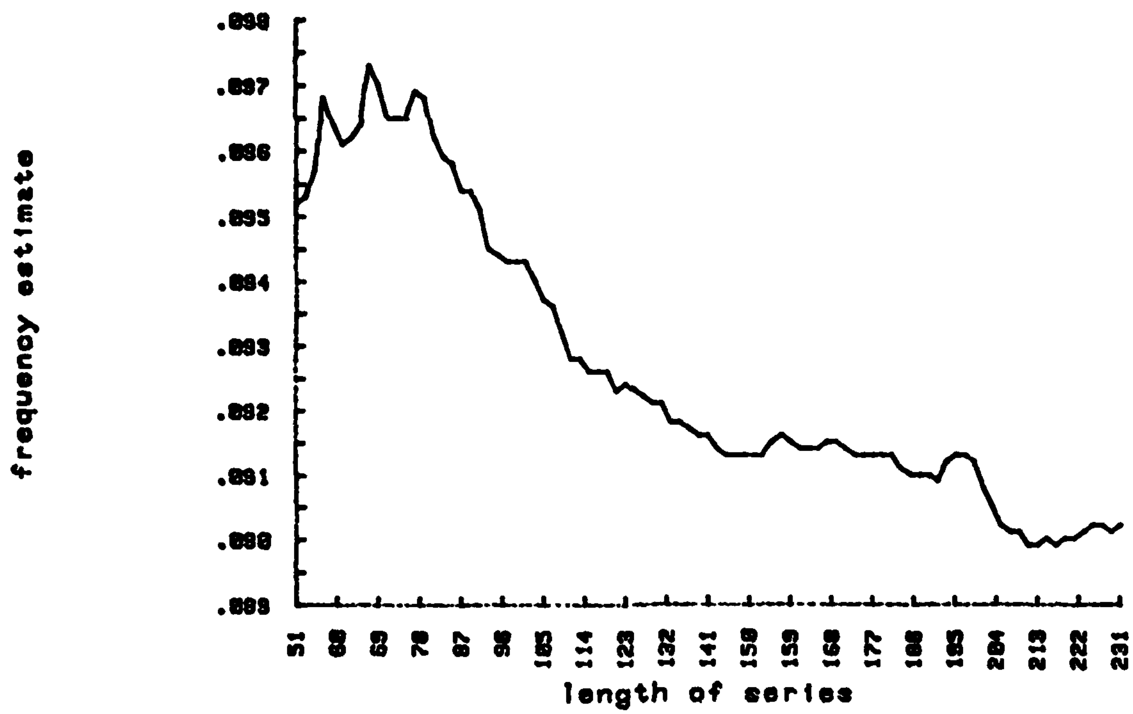

FIG. 6a. Reverse frequency trace of the 11.087-year period ($f_1^{(T)} = 0.0902$) in the annual sunspot numbers, 1749-1979, where $\Delta f = .0001$ and $T_{i+1}-T_i=2$, $i=1,2,\ldots,91$, $T_1=51$ (1929-1979).

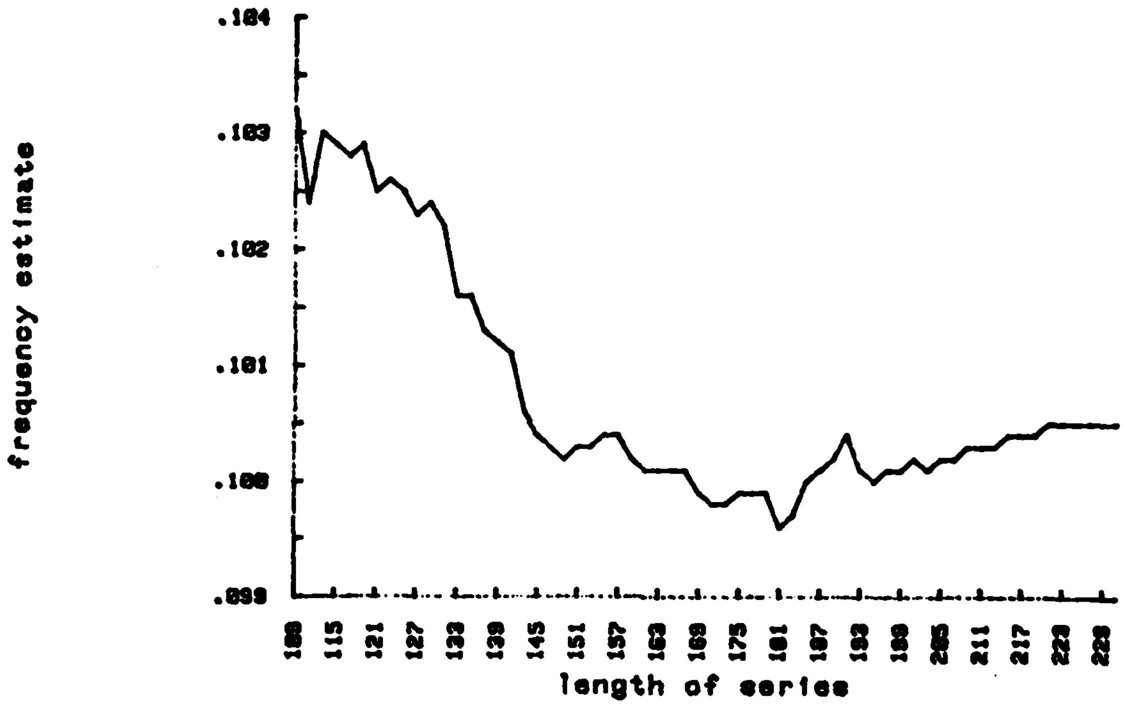

FIG. 6b. Reverse frequency trace of the 9.950-year period ($f_2^{(T)}$=0.1005) in the annual sunspot numbers, 1749-1979, where Δf=.0001 and $T_{i+1}-T_i=2$, $i=1,2,\ldots,62$, $T_1=109$ (1871-1979).

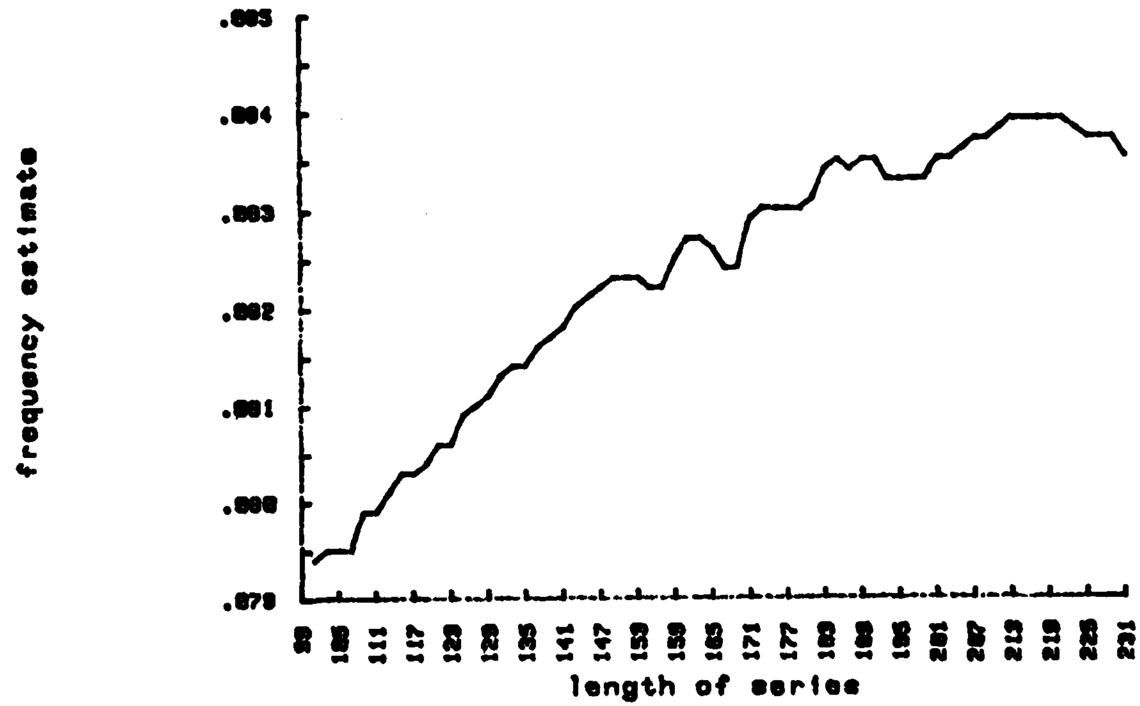

FIG. 6c. **Reverse** frequency trace of the 11.976-year period ($f_4^{(T)}$=0.0835) in the annual sunspot numbers, 1749-1979, where Δf=.0001 and $T_{i+1}-T_i=2$, $i=1,2,\ldots,66$, $T_1=101$ (1879-1979).

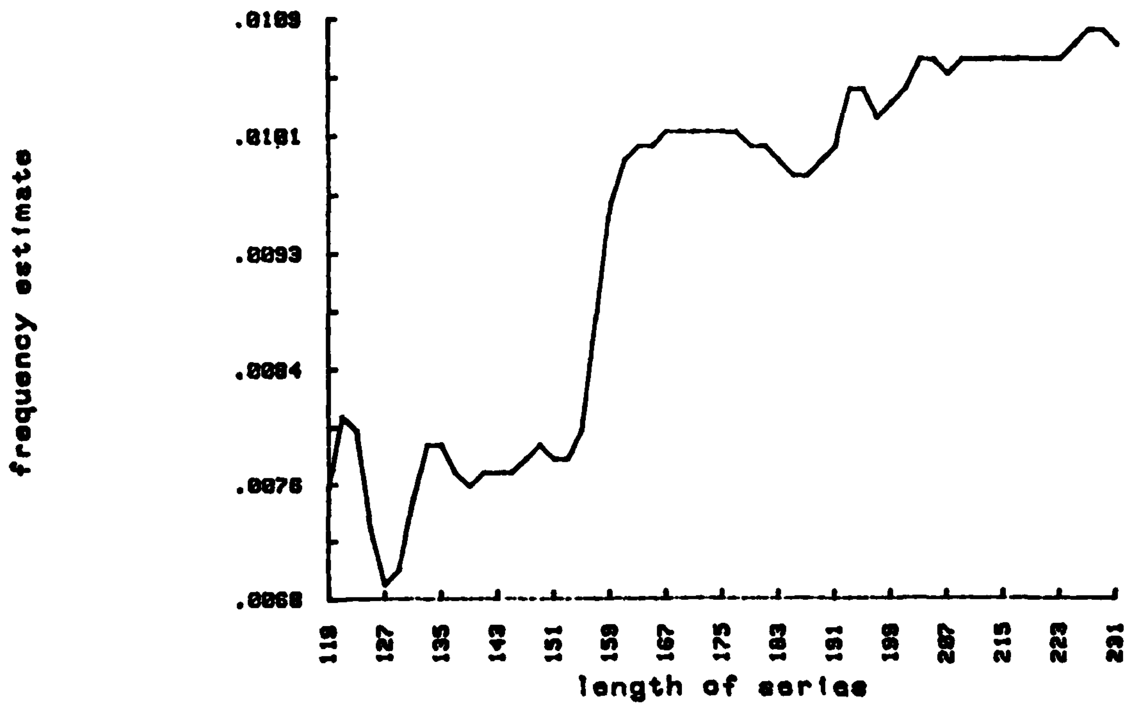

FIG. 7a. Reverse frequency trace of the 93.346-year period ($f_3^{(T)}$=.0107) in the annual sunspot numbers, 1749-1979, where Δf=.0001 and $T_{i+1}-T_i=2$, $i=1,2,\ldots,56$, $T_1=119$ (1861-1979).

FIG. 7b. Reverse frequency trace of the 58.140-year period ($f_6^{(T)}$=.0172) in the annual sunspot numbers, 1749-1979, where Δf=.0001 and $T_{i+1}-T_i=2$, $i=1,2,\ldots,44$. $T_1=143$ (1837-1979).

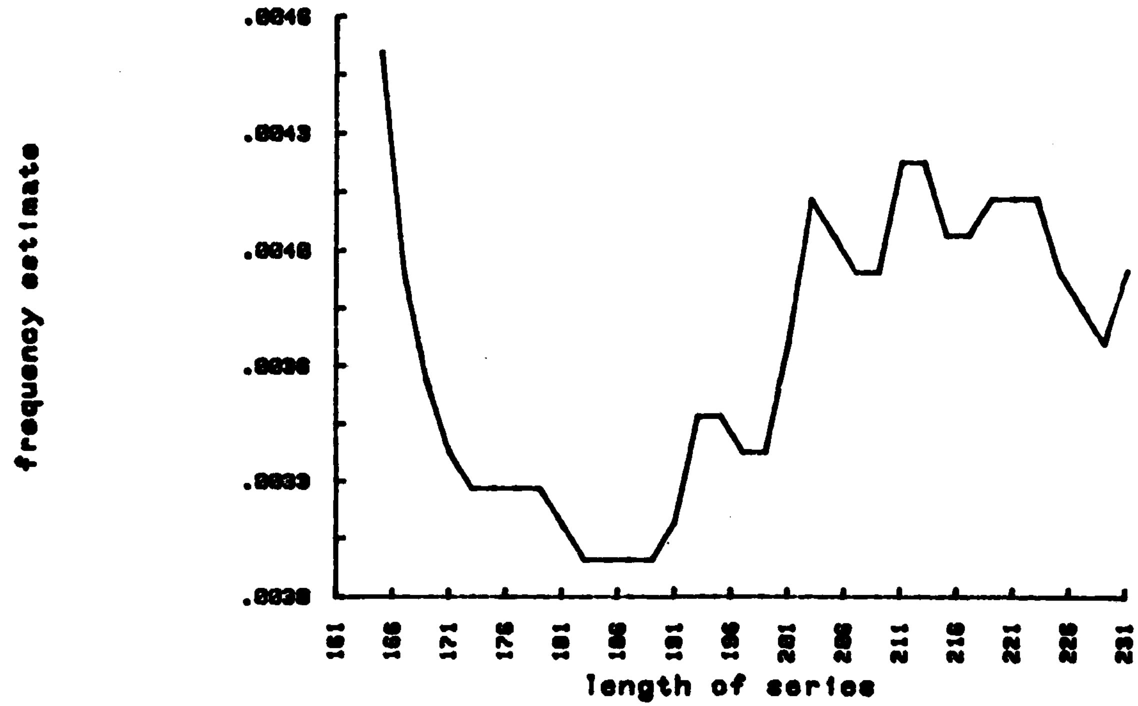

FIG. 7c. Reverse frequency trace of the 256.410-year period ($f_{11}^{(T)}$=.0039) in the annual sunspot numbers, 1749-1979, where Δf=.0001 and $T_{i+1}-T_i$=2, i=1,2,...,33, T_1=165 (1815-1979).

REFERENCES

AKAIKE, H. (1969), "Power Spectrum Estimation Through Autoregressive Model Fitting", Ann. Inst. Statist. Math., 21, 407-419.

ANDERSON, T. W. (1971). Time Series Analysis, New York: Wiley.

BERK, K. N. (1974), "Consistent Autoregressive Spectral Estimates", Ann. Statist., 2, 489-502.

BINGHAM, C., GODFREY, M. D., and TUKEY, J. W. (1967), "Modern Techniques in Power Spectrum Estimation", IEEE Trans. Audio Electroacoust., AU-15, 56-66.

BLACKMAN, R. B. and TUKEY, J. W. (1959), "The Measurement of Power Spectra, from the Point of View of Communications Engineering, New York: Dover.

BLOOMFIELD, P. (1976), Fourier Analysis of Time Series: An Introduction, New York: John Wiley.

BOLT, B. A. and CURRIE, R. G. (1975), "Maximum Entropy Estimates of Earth Torsional Eigenperiods from 1960 Trieste Data", Geophys. J. R. astr. Soc., 40, 107-114.

BOX, G. E. P. and JENKINS, G. M. (1976). Time Series Analysis, Forecasting and Control, Revised Edition. San Francisco: Holden-Day.

BRACEWELL, R. N. (1953), "The Sunspot Number Series", Nature, 171, 649-650.

BRAY, R. J. and LOUGHHEAD, R. E. (1965). Sunspots, New York: John Wiley.

BRILLINGER, D. R. (1975). Time Series: Data Analysis and Theory, New York: Holt, Rinehart and Winston.

BRILLINGER, D. R. and ROSENBLATT, M. (1967), "Asymptotic Theory of k-th Order Spectra" and "Computation and Interpretation of k-th Order Spectra", Spectral Analysis of Time Series, Ed. B. Harris, pp. 153-188 and 189-232 respectively. New York: Wiley.

BURG, J. P. (1968), "A New Analysis Technique for Time Series Data", paper presented at NATO Advanced Study Institute on Signal Processing, Enschede, Netherlands.

BURG, J. P. (1972), "The Relationship Between Maximum Entropy Spectra and Maximum Likelihood Spectra", Geophysics, 37, 375-376.

COGBURN, R. and DAVIS, H. T. (1974), "Periodic Splines and Spectral Estimation", Ann. Statist., 2, 1108-1126.

COOLEY, J. W. and TUKEY, J. W. (1965), "An Algorithm for the Machine Calculation of Complex Fourier Series", Math. Comp. 19, 297-301.

CRAMER, H. (1942), "On Harmonic Analysis in Certain Functional Spaces", Arkiv Math. Astr. Fysik., 28, 1-7.

CURRIE, R. G. (1980), "Detection of the 11-year Sunspot Cycle in Earth Rotation", Geophys. J. R. Astr. Soc., 61, 131-140.

DAMSLETH, E. and SPJØTVOLL, E. (1982), "Estimation of Trigonometric Components in Time Series", J. Amer. Statist. Assoc., 77, 381-387.

DANIELL, P. J. (1946), "Discussion of a Paper by M. S. Bartlett", J. Roy. Statist. Soc., Supplement, 8, 27.

DOOB, J. L. (1953). Stochastic Processes, New York: John Wiley.

EDDY, J. A. (1976), "The Maunder Minimum", Science, 192, 1189-1202.

FISHER, R. A. (1929), "Tests of Significance in Harmonic Analysis", Proceedings of the Royal Society, Series A, 125, 54-59.

FISHER, R. A. (1940), " On the Similarity of the Distributions Found for the Test of Significance in Harmonic Analysis, and in Stevens's Problem in Geometric Probability", Annals of Eugenics, 10, 14-17.

GOOD, I. J. (1958), "The Interaction Algorithm and Practical Fourier Analysis", J. Roy. Stat. Soc., Series B, 20, 361-372.

GOOD, I. J. and GASKINS, R. A. (1980), "Density Estimation and Bump-Hunting by the Penalized Likelihood Method Exemplified by Scattering and Meteorite Data" (with discussion), J. Amer. Statist. Assoc., 75, 42-73.

GRENANDER, U. and ROSENBLATT, M. (1957). Statistical Analysis of Stationary Time Series. New York: Wiley.

GUPTA, S. S. and SOBEL, M. (1962), "On the Smallest of Several Correlated F Statistics", Biometrika, 49, 509-523.

HANNAN, E. J. (1970). Multiple Time Series. New York: Wiley.

HANNAN, E. J. (1971), "Non-Linear Time Series Regression", J. Appl. Prob., 8, 767-780.

HANNAN, E. J. (1973), "The Estimation of Frequency", J. Appl. Prob., 10, 510-519.

HARTLEY, H. O. (1949), "Tests of Significance in Harmonic Analysis", Biometrika, , 194-201.

HERRING, R. W. (1980), "The Cause of Line Splitting in Burg Maximum-Entropy Spectral Analysis", IEEE Trans. Acoustics, Speech, and Signal Proc., ASSP-28, 692-701.

HILL, J. R. (1977), "Long-Term Solar Activity Forecasting Using High-Resolution Spectral Analysis", Nature, 266, 151-153.

IZENMAN, A. J. (1981), "J. R. Wolf and the Zürich Sunspot Relative Numbers", unpublished manuscript.

JENKINS, G. M. and WATTS, D. G. (1968). Spectrum Analysis and Its Applications, San Francisco: Holden-Day.

JENSEN, O. G. and ULRYCH, T. (1973), "An Analysis of the Perturbations on Barnard's Star", The Astronomical Journal, 78, 1104-1114.

JOHNSON, N. L. and KOTZ, S. (1972). Continuous Multivariate Distributions. New York: Wiley.

KARLIN, S. and RINOTT, Y. (1980), "Classes of Orderings of Measures and Related Correlation Inequalities. I. Multivariate Totally Positive Distributions", J. Multivariate Anal., 10, 467-498.

KIRK, B. L., RUST, B. W., and VAN WINKLE, W. (1979), "Time Series Analysis by the Maximum Entropy Method", Publication No. 1220, Environment Sciences Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee.

KOOPMANS, L. H. (1974). The Spectral Analysis of Time Series. New York: Academic Press.

KRISHNAIAH, P. R. and ARMITAGE, J. V. (1965), "Probability Integrals of the Multivariate F Distribution, with Tables and Applications", Aerospace Research Laboratories, Technical Report ARL 65-236, Wright-Patterson Air Force Base, Ohio.

KRISHNAIAH, P. R. and RAO, M. M. (1962), "Remarks on a Multivariate Gamma Distribution", American Mathematical Monthly, , 342-346.

LACOSS, R. T. (1971), "Data Adaptive Spectral Analysis Methods", Geophysics, 36, 661-675.

LANCZOS, C. (1956). Applied Analysis. Englewood Cliffs: Prentice Hall.

MALLER, R. A. and RESNICK, S. I. (1981), "Limiting Behavior of Sums and the Term of Maximum Modulus", unpublished manuscript.

MORAN, P. A. P. (1954), "Some Experiments on the Prediction of Sunspot Numbers", J. Royal Statist. Soc., , 112-117.

PARZEN, E. (1961), "Mathematical Considerations in the Estimation of Spectra", Technometrics, 3, 167-190.

PAUL, A. K. (1972), "Anharmonic Frequency Analysis", Mathematics of Computation, 26, 437-447.

PRIESTLEY, M. B. (1981). Spectral Analysis and Time Series, Vol. 1. Univariate Series. New York: Academic Press.

RUST, B. W. and KIRK, B. L. (1977), "Inductive Modelling of Population Time Series", In: Time Series and Ecological Processes, Proceedings of a S.I.M.S. Conference, Alta, Utah.

SCHUSTER, A. (1906), "On the Periodicities of Sunspots", Philos. Trans. Roy. Soc. London, Series A, 206, 69-100.

SHIMSHONI, M. (1971), "On Fisher's Test of Significance in Harmonic Analysis", Geophysical J. Roy. astr. Soc., 23, 373-377.

SIEGEL, A. F. (1980), "Testing for Periodicity in a Time Series", J. Amer. Statist. Assoc., 75, 345-348.

SIDDIQUI, M. M. and IZENMAN, A. J. (1981), "High-Resolution Frequency Analysis of the Monthly Sunspot Relative Numbers", unpublished manuscript.

STEVENS, W. L. (1939), "Solution to a Geometrical Problem in Probability", Annals of Eugenics, 9, 315-320.

ULRYCH, T. J. (1972), "Maximum Entropy Power Spectrum of Truncated Sinusoids", J. Geophys. Res., 77, 1396-1400.

ULRYCH, T. J. and BISHOP, T. N. (1975), "Maximum Entropy Spectral Analysis and Autoregressive Decomposition", Reviews of Geophysics and Space Sciences, 13, 183-200.

VAN DEN BOS, A. (1971), "Alternative Interpretation of Maximum Entropy Spectral Analysis", IEEE Trans. Inform. Theory, IT-17, 493-494.

WAHBA, G. (1979), "Automatic Smoothing of the Log Periodogram", J. Am. Statist. Assoc., 75, 122-132.

WALDMEIER, M. (1961). The Sunspot Activity in the Years 1610-1960, Zürich: Schulthess and Co.

WALKER, A. M. (1971), "On the Estimation of a Harmonic Component in a Time Series with Stationary Independent Residuals", Biometrika, 58, 21-36.

WALKER, A. M. (1973), "On the Estimation of a Harmonic Component in a Time Series with Stationary Dependent Residuals", Adv. Appl. Prob., 5, 217-241.

WELLS, F. J. and CHINNERY, M. A. (1973), "On the Separation of the Spectral Components of Polar Motion", Geophy. J. R. Astr. Soc., 34, 179-192.

WHITTLE, P. (1952), "The Simultaneous Estimation of a Time Series Harmonic Components and Covariance Structure", Trab. Estad., 3, 43-57.

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20. ABSTRACT Report No. 337

In this paper we describes the amplitude density function which is used to screen, identify and resolve frequencies in a time series. The technique is derived from the spectral representation of the sum of an harmonic regression function and a stochastic error process, and on the inversion theorem associated with that representation. The amplitude density function possesses the desirable properties of statistical consistency and high frequency resolution, and is related to the Fourier transform of a finite-length time series. Following the identification by the amplitude density function of the dominant frequencies in a time series, regression methods are used to fit a "hidden periodicities" model to the data. We Also introduce in this paper a high-resolution frequency trace to investigate the extent and direction of nonstationarity in a time series. These techniques are illustrated with a detailed frequency analysis of the annual sunspot numbers series from 1749 to 1979, and includes analysis of a physically motivated data transformation of the sunspot series.

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